UNIVERSITÀ DEGLI STUDI DI FIRENZE
Dipartimento di Sistemi e Informatica

Dottorato di Ricerca in
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XVI Ciclo

Robust Fault Detection and Isolation with a Sensitivity Constraint

Saverio Armeni

Ph.D. Coordinator
Prof. Edoardo Mosca

Advisors
Prof. Edoardo Mosca

Prof. Alessandro Casavola

Contents

Introduction

In recent years, control systems have become more complex and sophisticated. Consequently, the need of availability, reliability and operating safety is of increasing importance in order to avoid loss of performance or, even, critical system breakdowns. Since the early 70s, many authors have recognized the engineering relevance of the problem and have started to work to the design of diagnosis systems capable to detect fault or malfunction occurrences. A diagnosis system is also called fault detection and identification (FDI) system. The most common approach to FDI is hardware redundancy which is the direct comparison of outputs from identical components. However, this approach is expensive, limited by space and weight and subject to failure too. An alternative is analytical redundancy which uses a mathematical model in conjunction with inputs and outputs of system in order to generate a diagnostic signal. This signal should be generated in such a way to be non-zero only when a fault has occurred and insensitive as much as possible to other exogenous signals. In the following, the diagnostic signal will be referred to also as residual. Within this approach no redundant components are needed, only additional computation is required. A popular approach to analytical redundancy is the detection filter which was first introduced by Beard and refined by Jones. It is also known as Beard-Jones detection (BJD) filter. The BJD filter is designed in such a way the direction of the residual can be used to identify a specific fault occurrence. In [Mas86] and [MVW89], a comprehensive geometric interpretation of BJD filter is carried out by M. A. Massoumnia et al. Other solutions to the detection filter design can be found in the works of J. L. Speyer et al. [WS87] and [CS98]. Moreover, the design of detection filters for nonlinear systems is addressed by A. Isidori et al. in [PI01]. However, in order to face modeling uncertainties many authors have considered the case of designing robust model-based detection filters, as stated by A. Edelmayer et al. [EBSK97] and J. L. Speyer et al. [CS02]. An extensive survey on the various robust model-based FDI approaches can be found in the book of J. Chen and R. J. Patton, [CP99]. Another approach to such problems is the use of learning systems and neural networks in FDI, as stated by M. Polycarpou et al. in [PH95]. Statistical approaches were also developed, in parallel with the development of the BJD filter, starting in the early 70s. The fault diagnosis is performed by statistical testing on whiteness, mean and covariance of residuals. A survey on the statistical approach can be found in the book of M. Basseville and I. V. Nikiforov, [BN93]. This thesis addresses the problems of designing robust model-based detection filters for systems subject to disturbances and model uncertainties. In particular, it is focused on the problem of designing detection filters with enhanced fault detection capability.

Thesis Outline

The thesis is centered on the problem of designing a FDI filter with enhanced fault sensitivity, i.e. the transmission from one fault is maximized while the transmission from other faults, denoted as nuisances, is minimized. Moreover, the transmission from process and measurement noises is minimized too, in such a way that the filter is robust with respect to these disturbances. A new fault detection and isolation algorithm is determined. The algorithm is obtained solving a family of LMI optimization problems. The thesis consists of two parts. The first is an introductory part, where some basic notions are given. The second part explains the major results of the research activity.
Chapter 1 gives an introduction to the fault detection and identification, where some related definitions are given. Furthermore, the model-based approach is explained in detail through an analytical example. Chapter 2 gives an introduction of the linear matrix inequality optimization method. In particular some useful tools related to the control theory are provided. Chapter 3 describes

some dynamical system properties and some geometric aspects that will be useful in the second part. Chapter 4 introduces an FDI design method for linear time-invariant (LTI) systems subject to disturbances. Hence, an eigen-structure assignment is used to enhance the fault detection capability of the filter. In conclusion a new algorithm for designing FDI filters is introduced. This algorithm is derived from solving a family of LMI optimization problems. A simple designing example ends the chapter. Chapter 5 considers the case of designing an FDI filter for a class of systems subject to model uncertainty. Chapter 6 considers the case of designing an FDI filter for linear parametric variable (LPV) systems. The worth of this class of systems is that some nonlinear system can be described through a quasi-LPV model. Hence, a FDI filter designed for LPV systems can also be used in the FDI of nonlinear system. A designing example ends the chapter. Chapter 7 consists of comparisons with other methods. The main results obtained are briefly summarized in the conclusions, at the end of the thesis.

Part I

Basic concepts

Chapter 1

Fault Detection and Identification

This chapter introduces some basic notions about the fault detection over controlled systems. Consider a system with its physical plant, sensors and actuators. All these components are subject to failure, malfunction or simply something that can cause a deflection in the correct system behavior. In the following, the term *system fault* refers to this system malfunction. In particular, a fault could be a malfunction in the sensors or actuators and, even, a change in the plant parameters. The correct diagnosis of a system fault can be helpful to avoid critical system breakdown. Therefore, a monitoring system that is able to detect and to diagnose a fault occurrence in the system is called a *fault diagnosis system*. For more details on the arguments exposed in this chapter see [CP99].

1.1 Fault diagnosis system

A fault diagnosis system normally consists of the following tasks:

fault detection is the detection of a system fault and its starting time;

fault isolation is the determination of the kind, location and time of the faults;

fault identification is the determination of the size and the time behavior of the fault.

In practice, the most frequently used diagnosis method is to monitor the level of a particular signal, and take action when the signal reaches a given threshold. This method, although simple to implement has serious drawbacks. The first drawback is the possibility of false alarms in the event of noise, input variations and change of operating point. The second drawback is that a single fault could cause many system signals to exceed their limits and appear as multiple faults, and hence fault isolation is very difficult. Indeed, the use of a consistency checking for a number of system signals which can eliminate the above problems, is an important way of enhancing the detection and isolation of fault diagnosis capability of an automated system. However, a mathematical model which gives functional relationship among different system signals is needed. A traditional approach to the fault diagnosis is based on the *hardware redundancy*. This method is based on the replications of the critical system components, i.e. sensors and actuators, then a consistency test between them can be used to detect a fault occurrence. The major drawbacks with hardware redundancy are the extra equipment and maintenance cost, and, furthermore, the additional space to accommodate the equipment. To avoid this troubles the *analytical redundancy* approach can be useful. With analytical redundancy, instead of replicating the physical component of the system, a mathematical model is used to describe the relationship between the measured variables of the system and the faults. Moreover, no additional hardware faults are introduced into an analytical redundant scheme, because no extra hardware is required, hence analytical redundancy is potentially more reliable than hardware redundancy. In analytical redundancy schemes, the resulting difference generated from the consistency checking of different variables is called *residual signal*. The residual should be zero-valued when the system is normal, and should diverge from zero when a fault occurs in the system. Analytical redundancy makes use of a mathematical model of the monitored process and is therefore often referred to as the *model-based approach* to fault diagnosis. Model-based FDI makes use of mathematical models of the supervised system. However a perfect mathematical model of a physical system is never available. Usually, the parameters of the system may vary with time in an uncertain manner, and the characteristics of the disturbances and noise are unknown so that they cannot be modeled accurately. Hence, there is always a mismatch between the actual process and its mathematical model even if there are no process faults. To overcome the difficulties introduced by modeling uncertainty a model-based fault detection and isolation has to be made robust, i.e. insensitive to modeling uncertainty, disturbance and noise.

1.2 The Model-based FDI

Model-based fault diagnosis can be defined as the detection, isolation and characterization of faults in components of a system from the comparison of the system's available measurements, with a priori information represented by the system's mathematical model. The fault detection is carried out with a two-stages structure. The two main stages are described as follows:

Residual Generation: Its purpose is to generate a fault indicating signal (the residual), using available input and output information from the monitored system. The residual should be normally zero or close to zero when no fault is present, but is distinguishably different from zero when a fault occurs.

Decision Making: The residuals are examined for the likelihood of faults, and a decision rule is then applied to determine if any faults have occurred. A decision process may consist of a simple threshold test on the instantaneous values of the residuals, or it may consist of methods of statistical decision theory, e.g., generalized likelihood ratio (GLR) testing or sequential probability ratio testing (SPRT).

Model-based FDI is concerned mainly with *on-line* fault diagnosis, in which the diagnosis is carried out during system operation. The system model required in model-based FDI is the open-loop system model although the system is in the control loop. Hence, the information used for FDI is the measured output from sensors and the input to the actuators. This is because the input and output information required in model-based FDI is related to the open-loop system.

Hence, it is not necessary to consider the controller in the design of a fault diagnosis scheme. This is consistent with the separation principle in control theory because fault diagnosis can be broadly treated as a special observation problem. In the case when the input to the actuator is not available, the only choice is to use the reference command and the measured output, i.e. the closed-loop model. For those cases, the controller plays an important role in the design of diagnostic schemes. Because, a robust controller may desensitize fault effects and make the diagnosis very difficult. In such case, the best solution is to design the FDI scheme and the controller simultaneously. To give a generic idea of the model-based FDI, let consider a dynamical system with sensor and actuator faults, subject to disturbances as depicted in Fig. 1.1. For simplicity assume

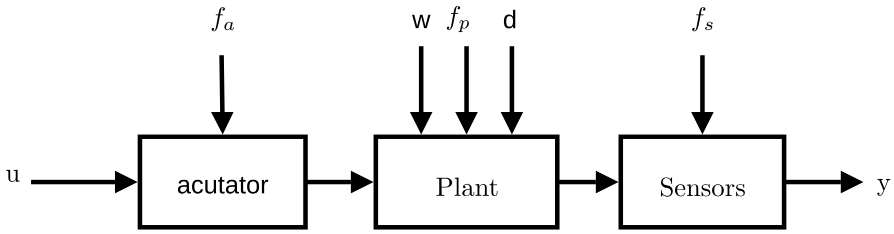


Figure 1.1. Plant

that the mathematical model is represented by the following dynamical system

$$\begin{cases} \dot{x}(t) &= (A + \Delta A f_p(t))x(t) + Bu(u) + B_d d(t) + L f_a(t) \\ y(t) &= Cx(t) + Du(t) + E f_s(t) + D_d d(t) \end{cases} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ indicates the state-to-state matrix, $\Delta A \in \mathbb{R}^{n\times n}$ the state parametric change matrix, $B \in \mathbb{R}^{n\times m}$ the input-to-state matrix, $C \in \mathbb{R}^{p\times n}$ the state-to-output matrix, $D \in \mathbb{R}^{p\times m}$ the input-to-output matrix, $L \in \mathbb{R}^{n\times q}$ the fault-to-state matrix, $E \in \mathbb{R}^{p\times q}$ the fault-to-output-state matrix, $B_d \in \mathbb{R}^{n\times k}$ the disturbance-to-state matrix and $D_d \in \mathbb{R}^{p\times q}$ the disturbance-to output matrix. The vectors $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $d \in \mathbb{R}^k$ indicate, respectively, the state, the input, the output and the disturbance vector, whilst $f_a \in \mathbb{R}^q$, $f_s \in \mathbb{R}^k$ and $f_p \in \mathbb{R}^k$ indicate, respectively, the sensor, the actuator and the parametric fault vectors. Recall that both the disturbance vector and the fault vectors are all unknown vector signals. In (1.1), the parametric faults, which correspond to a change in the system parameters are modeled in a multiplicative way, hence $f_p(t)$ is said to be a **multiplicative fault**. While, the sensor and actuator faults are modeled in an additive way, hence $f_a(t)$ and $f_s(t)$ are said to be **additive faults**. As example, consider a fault in the i-th actuator

$$B_i(u_i(t) + f_{a_i}(t)) \tag{1.2}$$

where B_i is the i-th column of the matrix B, related to the i-th entry $u_i(t)$ of the input vector $u(t)$ and $f_i(t)$ indicates the fault signal. In order to model such actuator fault, the matrix L should have a column such that $L_i = B_i$, related to the i-th entry f_{a_i} of the fault vector $f_a(t)$. Moreover, in order to consider the actuator failure, the signal f_{a_i} can be modeled as $f_{a_i}(t) = -\beta(t - t_f)u_i(t)$ where $\beta(t-t_f)$ represents the time-behavior of the fault and t_f is the start-time of the fault action. The time-behavior of the fault can be **abrupt** or **incipient**. With abrupt it is meant a non-smooth time-behavior like the following

$$\beta(t - t_f) = \begin{cases} \delta & t \geq t_f \\ 0 & elsewhere \end{cases} \tag{1.3}$$

whilst incipient means a smooth time-behavior like the following

$$\beta(t - t_f) = \begin{cases} \delta(1 - e^{-\alpha(t-t_f)}) & t \geq t_f \\ 0 & elsewhere \end{cases} \tag{1.4}$$

with δ the fault intensity and α a positive real decay rate. In the same way it is possible to model a sensor fault. Suppose that there is a fault in the i-th sensor then

$$y_i(t) = C_i(x_i(t) + f_{s_i}(t)) \tag{1.5}$$

hence, $E_i = C_i$ and $f_{s_i}(t) = -\beta(t - t_f)x_i(t)$. In order to use the same design methods for both sensor and actuator faults, the sensor fault can be modeled as pseudo-actuator fault. Hence, consider a generic sensor fault f_s described by

$$y(t) = Cx(t) + Ef_s(t) \tag{1.6}$$

if a matrix $F_{pa} \in \mathbb{R}^{n \times q}$ such that

$$CF_{pa} = E \tag{1.7}$$

exists, then

$$y(t) = C(x(t) + F_{pa}f_s(t)) \tag{1.8}$$

It is possible to introduce the following new state variable

$$z(t) = x(t) + F_{pa} f_s(t) \tag{1.9}$$

then, differentiating (1.9) gives

$$\begin{cases} \dot{z}(t) &= Az(t) + F_{pa}\dot{f}_s(t) - AF_{pa}f_s(t) \\ y(t) &= Cz(t) \end{cases} \tag{1.10}$$

In conclusion the sensor fault map E can be replaced by the pseudo-actuator fault map $[-AF_{pa} \quad F_{pa}]$ in the new state variable $z(t)$. In such a way it is possible to use the same design methods, for both sensor and actuator faults.

1.3 Model-based Fault Detection Filters

In practice, the most frequently used FDI approach uses information known a priori about the characteristics of certain signals (e.g. amplitude and frequency properties). The main shortcomings of this group of methods is the necessity to have a pattern of the signal characteristics, the unavoidable dependence of these characteristics on operating states of the system, which are not exactly known a priori. The Model-based approach to FDI consists in the design of a residual generator filter which is capable to determine the fault occurrence from the input and output signals of the plant. To this end it is necessary the knowledge of a mathematical model of the plant. Residuals are quantities that represent the inconsistency between the actual system variables and the mathematical model. However, the mathematical model can be approximated and the plant can be subject to disturbances. Then, in such cases it is necessary to design residual generators that will be robust to model uncertainties and disturbances. Consider the input-output representation of (1.1)

$$y(s) = G(s)u(s) + G_f(s)f(s) + G_d(s)d(s) \tag{1.11}$$

with

$$G(s) = C(sI - A)^{-1}B + D \tag{1.12}$$

$$G_f(s) = C(sI - A)^{-1}L + E \tag{1.13}$$

$$G_d(s) = C(sI - A)^{-1}B_d + D_d \tag{1.14}$$

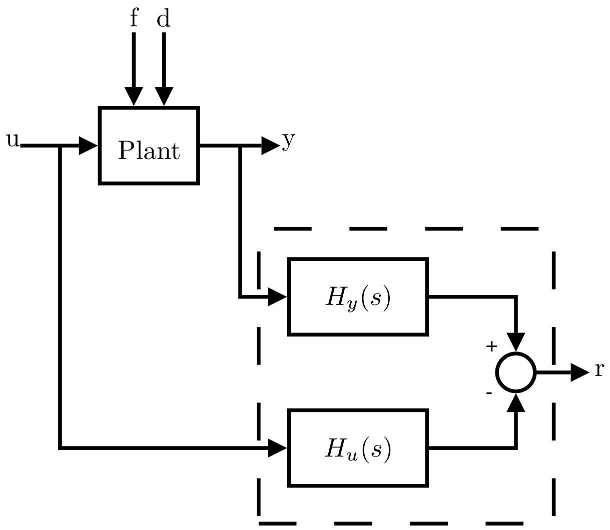


Figure 1.2. General structure for a residual generator

Then, the residual generator can be defined as

$$\begin{aligned} r(s) &= H_y(s)y(s) - H_u(s)u(s) \\ &= (H_y(s)G(s) - H_u(s))u(s) + H_y(s)G_f(s)f(s) + H_y(s)G_d(s)d(s) \end{aligned} \tag{1.15}$$

with $H_y(s)$ and $H_u(s)$ two proper transfer functions to be determined. Fig. 1.2 shows a general structure for the residual generator. However, the residual $r(s)$ must satisfy the following property

$$\begin{cases} r(s) \equiv 0 & \textit{if } f(s) \equiv 0 \\ r(s) \neq 0 & \textit{if } f(s) \neq 0 \end{cases} \tag{1.16}$$

irrespective to $y(s)$, $u(s)$ and $d(s)$. Then, the residual generator problem is to design $H_y(s)$ and $H_u(s)$ in order to satisfy (1.16) and maintaining the stability of the residual generator. Moreover in the case of multiple simultaneous faults, the residual generator will be able to isolate each fault occurrence. Then to guarantee the condition that the residual is null for a non-faulty condition, the following condition must be satisfied by $H_y(s)$ and $H_u(s)$

$$H_y(s)G_d(s) \equiv 0 \tag{1.17}$$

$$H_y(s)G(s) - H_u(s) \equiv 0 \tag{1.18}$$

Moreover, in the case of multiple simultaneous faults the transfer function $H_y(s)G_f(s)$ must be diagonal. A way to design $H_y(s)$ and $H_u(s)$ can be the use of observer-based approach. The basic idea is to estimate the outputs of the system from the measurements by using either Luenberger observer. Then, the output estimation error is used as a residual. Consider the following full observer structure

$$\begin{cases} \dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + (B - KD)u(t) + Ky(t) \\ \hat{y}(t) = C\hat{x}(t) + Du(t) \end{cases} \tag{1.19}$$

The residual can be expressed as the weighted output estimation error

$$r(t) = Q(y(t) - \hat{y}(t)) \tag{1.20}$$

Then

$$H_y(s) = Q\left(I - C(sI - (A - KC))^{-1}K\right) \tag{1.21}$$

$$H_u(s) = Q\left(C(sI - (A - KC))^{-1}(B - KD) + D\right) \tag{1.22}$$

In conclusion, the matrices K and Q will be designed in such a way to guarantee the condition (1.16). Chapters 4, 5 and 6 carry out the problem of designing these pairs of matrices in order to design residual generators, capable of detecting and isolating the fault signals of the supervised system.

1.4 Conclusions

This chapter has introduced the concept of fault diagnosis system. The differences between the hardware redundancy and the analytical redundancy methods are explained. Then, the model-based FDI is presented. A brief fault modeling is presented. In conclusion, a simple model-based FDI filter is analyzed.

Chapter 2

Linear Matrix Inequalities

The linear matrix inequalities (LMI's) have emerged as a powerful tool to approach control problems in the past two decades. In particular, powerful numerical interior point techniques have been developed to solve LMI's in a practically efficient manner. Many authors have worked in order to introduce a comprehensive control theory based on LMI. Hence, many control problems have been revised in order to be solved through these optimization methods. For more details about the arguments exposed in this chapter see [BGFB94].

2.1 Preliminary definitions

A linear matrix inequality is an expression of the form:

$$F(x) \doteq F_0 + x_1 F_1 + \cdots + x_m F_m > 0 \tag{2.1}$$

with $x = (x_1, \ldots, x_m)$ is a vector of m real numbers called the decision variables, $F_0, \ldots, F_m$ are real symmetric matrices, i.e. $F_i = F_i' \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, m$ for some $n \in \mathbb{Z}_+$. The inequality > 0 in (2.1) means positive definite, i.e. $u'F(x)u > 0$ for all $u \in \mathbb{R}^n$, $u \neq 0$, or equivalently the smallest eigenvalue of $F(x)$ is positive.

Definition 1 ***Linear Matrix Inequality*** *A linear matrix inequality (LMI) is an inequality*

$$F(x) > 0 \tag{2.2}$$

where F is an affine function mapping a finite dimensional vector space $\mathbb{V}$ to the set $\mathbb{S} \doteq \{M | \exists n > 0 \quad such \quad that \quad M = M' \in \mathbb{R}^{n \times n}\}$, of real symmetric matrices.

Recall that $F(x)$ affine in x means that

$$F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y) \quad \forall \lambda \in \mathbb{R} \tag{2.3}$$

Moreover, the linear matrix inequality defines a *convex constraint* on x. That is the set

$$\mathcal{C} \doteq \{x | F(x) > 0\} \tag{2.4}$$

is convex. Indeed, if $x, y \in \mathcal{C}$ and $\lambda \in (0,1)$ then

$$F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y) > 0 \tag{2.5}$$

where the last inequality follows from the fact that $\lambda \geq 0$ and $(1-\lambda) \geq 0$. Then, the linear matrix inequality $F(x) > 0$ defines a *convex constraint* on the variable x. The optimization problems involving the minimization (or maximization) of a cost functional $f : \mathcal{C} \longrightarrow R$ with $\mathcal{C} \doteq \{x | F(x) > 0\}$ belong to the class of convex optimization problems. If the cost functional f is a convex function then the function has only global minimum (maximum) points, therefore it is possible to use the full power of convex optimization theory.
Suppose that F, G, $H : \mathbb{V} \longrightarrow \mathbb{S}$ are affine functions. There are three generic problems related to the study of linear matrix inequalities:

1. **Feasibility:** The test whether or not, there exist solutions $x \in \mathbb{V}$ of $F(x) > 0$ is called *feasibility problem.*

2. **Optimization:** Let $f : \mathcal{C} \longrightarrow R$ and suppose that $\mathcal{C} = \{x | F(x) > 0\}$. The problem to determine

$$V_{opt} = \inf_{x \in \mathcal{C}} f(x) \tag{2.6}$$

   is called an optimization problem with an LMI constraint

3. **Generalized eigenvalue problem:** This problem amounts to minimizing a scalar $\lambda \in \mathbb{R}$ subject to

$$\begin{cases} \lambda F(x) - G(x) > 0 \\ F(x) > 0 \\ H(x) > 0 \end{cases} \tag{2.7}$$

2.2 Useful tools

The *Schur complement* is a powerful tool which can be used to transform a nonlinear inequality into a linear one. Suppose that the matrix $M \in \mathbb{R}^{n \times n}$ is partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{2.8}$$

where M_{11} has dimension $r \times r$. Assume that M_{11} is non-singular. Then the matrix

$$S \doteq M_{22} - M_{21} M_{11}^{-1} M_{12} \tag{2.9}$$

is called the *Schur complement* of M_{11} in M. If M is symmetric then

$$M > 0 \iff \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} > 0 \iff \begin{cases} M_{11} > 0 \\ S > 0 \end{cases} \tag{2.10}$$

Lemma 1 ***Schur complement*** *Let $F : \mathbb{V} \longrightarrow \mathbb{S}$ be an affine function which is partitioned according to*

$$F(x) = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix} \tag{2.11}$$

where $F_{11}(x)$ is square. Then $F(x) > 0$ if and only if

$$\begin{cases} F_{11}(x) > 0 \\ F_{22}(x) - F_{12}(x)[F_{11}(x)]^{-1} F_{21}(x) > 0 \end{cases} \tag{2.12}$$

or equivalently if it is supposed that F_{22} is square, then $F(x) > 0$ if and only if

$$\begin{cases} F_{22}(x) > 0 \\ F_{11}(x) - F_{21}(x)[F_{22}(x)]^{-1} F_{12}(x) > 0 \end{cases} \tag{2.13}$$

Note that the second inequality in (2.12) and in (2.13) is a non-linear matrix inequality in x. Using this result, it follows that non-linear matrix inequalities of the form (2.12) or (2.13) can be converted to linear matrix inequalities. In order to introduce the *Bounded Real Lemma* some preliminary definitions are needed. Consider a continuous time-invariant linear system Σ described by the equations

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{2.14}$$

with $\mathcal{X} \subset \mathbb{R}^n$ the state space, $\mathcal{U} \subset \mathbb{R}^m$ the input space and $\mathcal{Y} \subset \mathbb{R}^p$ the output space. Let $s : \mathcal{U} \times \mathcal{Y} \longrightarrow \mathbb{R}$ be a quadratic supply function defined by

$$s(u, y) = \begin{pmatrix} y \\ u \end{pmatrix}' \begin{pmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu} \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} \tag{2.15}$$

Definition 2 ***Dissipativity*** *The system Σ with supply rate $s(u, y)$ is said to be dissipative if there exists a non-negative function $V : X \longrightarrow \mathbb{R}$ such that*

$$V(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt \geq V(x(t_1)) \tag{2.16}$$

for all $t_0 \leq t_1$ and all trajectories (u, x, y) which satisfy (2.14) □

Theorem 1 *Suppose that the system Σ described by (2.14) is controllable and let the supply function s be defined by (2.15). Then the following statements are equivalent.*

1. *(Σ, s) is dissipative.*

2. *(Σ, s) admits a quadratic storage function $V(x) \doteq x'Px$ with $P = P' \geq 0$.*

3. *There exists $P = P' \geq 0$ such that*

$$-\begin{pmatrix} A'P + PA & PB \\ * & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}' \begin{pmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu} \end{pmatrix} \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \geq 0 \tag{2.17}$$

4. *For all $\omega \in \mathbb{R}$ with $det(i\omega I - A) \neq 0$, $G(s) \doteq C(sI - A)^{-1}B + D$ satisfies*

$$\begin{pmatrix} G(i\omega) \\ I \end{pmatrix}^* \begin{pmatrix} Q_{yy} & Q_{yu} \\ Q_{uy} & Q_{uu} \end{pmatrix} \begin{pmatrix} G(i\omega) \\ I \end{pmatrix} \geq 0 \tag{2.18}$$

□

Consider the following supply function

$$s(u, y) = \gamma^2 u'u - y'y \tag{2.19}$$

where $\gamma \geq 0$, from Theorem 1 the following result is obtained:

Lemma 2 ***Bounded Real Lemma*** *Suppose that the system Σ (2.14) is controllable and has the transfer function G. Let $s(u, y) = \gamma^2 u'u - y'y$ be a supply function. The following statements are equivalent*

1. *(Σ, s) is dissipative*
2. *The system of LMI's*

$$\begin{aligned} \begin{pmatrix} A'P + PA + C'C & PB + C'D \\ * & D'D - \gamma^2 I \end{pmatrix} &\leq 0 \\ P = P' &\geq 0 \end{aligned} \tag{2.20}$$

   is feasible.

3. *The system of LMI's*

$$\begin{aligned} \begin{pmatrix} A'P + PA & PB & C' \\ * & -\gamma & D' \\ * & * & -\gamma \end{pmatrix} &\leq 0 \\ P = P' &\geq 0 \end{aligned} \tag{2.21}$$

   is feasible.

4. *For all $\omega \in \mathbb{R}$ with $det(sI - A) \neq 0$, $G(i\omega)^* G(i\omega) \leq \gamma^2 I$.*

□

This lemma would be more useful in the $\mathcal{H}_\infty$ design. Moreover, all the signals $y(t)$ and $u(t)$ are assumed to belong to the $\mathcal{L}_2$ functional space, defined by

Definition 3 ***$\mathcal{L}_2$ space*** *The $\mathcal{L}_2$ is a Hilbert space of functions over $\mathbb{R}$ consisting of all functions $f(t)$ such that the integral below is bounded*

$$\int_0^\infty f'(t)f(t)dt < \infty \tag{2.22}$$

The inner product for this Hilbert space is defined as

$$\langle f, g \rangle \doteq \int_0^\infty f'(t)g(t)dt \tag{2.23}$$

for $f(t)$, $g(t) \in \mathcal{L}_2$ and the inner product induced norm is given by

$$\|f(t)\|_2 \doteq \sqrt{\langle f, g \rangle} \tag{2.24}$$

□

Moreover, the use of signals that belong to the $\mathcal{L}_2$ space is not restrictive, because all the results that are valid for a signal that belongs to the $\mathcal{L}_{2T}$ space can be extended to a signal that belongs to the $\mathcal{L}_{2T}$ space, which is defined as

Definition 4 ***$\mathcal{L}_{2T}$ space*** *The $\mathcal{L}_{2T}$ space is a Hilbert space of functions over $\mathbb{R}$ consisting of all functions $f(t)$ such that the integral below is bounded*

$$\int_0^T f'(t)f(t)dt < \infty \tag{2.25}$$

with $T > 0$ a real constant. □

Another useful tool for design is the possibility to assign the pole of a filter in a prescribed region of the complex plane. This can be done with the introduction of *LMI regions.*

Definition 5 ***LMI region*** *An LMI region is any subset $\mathcal{D}$ of the complex plane $\mathbb{C}$ defined as*

$$\mathcal{D} = \{z \in \mathbb{C} | L + zM + \bar{z}M' < 0\} \tag{2.26}$$

with $L = L'$ and M real matrices.

The matrix-valued function

$$f_{\mathcal{D}}(z) = L + zM + \bar{z}M' \tag{2.27}$$

is called the characteristic function of $\mathcal{D}$. Below are a few examples of LMI regions:

- left half-plane with $Re(z) < -\alpha$: $f_{\mathcal{D}}(z) = z + \bar{z} + 2\alpha < 0$;

- disk centered at $(-q, 0)$ with radius r:

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -r & q+z \\ q+\bar{z} & -r \end{bmatrix} < 0$$

- conic sector with apex in β and inner angle 2θ:

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -2\beta\cos\theta + \sin\theta(z+\bar{z}) & \cos\theta(z-\bar{z}) \\ \cos\theta(\bar{z}-z) & -2\beta\cos\theta + \sin\theta(z+\bar{z}) \end{bmatrix} < 0$$

Notice that

- the intersection of LMI regions are also LMI regions;
- any convex region symmetric respect to the real axis can be approximated by an LMI region to any desired accuracy;

Definition 6 ***$\mathcal{D}$-stability*** *A matrix $A \in \mathbb{R}^{n\times n}$ is $\mathcal{D}$-stable, i.e. all its eigenvalues are in the LMI region $\mathcal{D}$, i.e. if and only if there exists a positive definite matrix $X = X'$ such that*

$$M_{\mathcal{D}}(A, X) \doteq L \otimes X + M \otimes (XA) + M' \otimes (XA)' < 0 \tag{2.28}$$

This result can be seen as a generalization of the Lyapunov theorem because for the usual stability region $f_{\mathcal{D}}(z) = z + \bar{z} < 0$, (2.28) reduces to

$$1 \otimes (XA) + 1' \otimes (XA)' = XA + (XA)' < 0$$

2.3 Conclusions

This chapter has introduced the linear matrix inequalities and other related useful tools.

Chapter 3

Dynamical System Properties

This chapter explains some geometric aspects of dynamical system properties which can be useful in the design of fault detection filters. The geometric approach is a control theory for multi-variable linear systems based on vector spaces and subspaces. It is also based on linear transformations. Some geometric properties about linear time-invariant systems are introduced. Furthermore, a class of affine dynamical systems is presented.

3.1 Preliminary definitions

It is useful to recall the properties of reachability and unobservability subspaces for LTI systems. To this end, consider the following system

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{3.1}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$. The reachability subspace $\mathcal{R}$ of (3.1) is defined as the set of all the states x that can be reached in any finite time by means of a control action u and is defined by

$$\mathcal{R} \doteq \mathcal{I}(B) + A\mathcal{I}(B) + \cdots + A^{n-1}\mathcal{I}(B) \tag{3.2}$$

where $\mathcal{I}(B)$ denotes the subspace generated by the columns of the matrix B. While the unobservability subspace $\mathcal{U}$ of (3.1) is the set of all initial states that cannot be recognized from the output functions. It is defined by

$$\mathcal{U} \doteq \mathcal{N}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) \tag{3.3}$$

Moreover, if A is invertible, it is defined also by

$$\mathcal{U} \doteq \mathcal{N}(C) \cap A^{-1}\mathcal{N}(C) \cap \cdots \cap A^{-n+1}\mathcal{N}(C) \tag{3.4}$$

where $\mathcal{N}(C)$ denotes the null space generated by the rows of the matrix C.
The fault detection filters are designed in such a way to detect a fault occurrence by filtering the input and output signals of the faulty system. From a practical point of view a fault signal is a particular unknown input, that needs to be detected when its value is different from zero. Hence, it is necessary that the fault occurrence is *observable* from the outputs. In general, an input is said to be *input observable* if it is possible to deduce from the outputs if its value differs from zero.

Definition 7 *[SM69]**Input Observability** Consider (3.1), the input $u(t)$ is said to be observable from the output $y(t)$ if*

$$y(t) = 0 \ for \ t \geq 0 \Longrightarrow u(t) = 0 \ for \ t > 0 \tag{3.5}$$

provided that $x(0) = 0$. □

The next lemma introduces the terms such that the generic input u of a system like (3.1) is *input observable.*

Lemma 3 *[SM69] A system like (3.1) is said to be input observable if and only if*

$$rank \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{n-1}B \end{bmatrix} = m \tag{3.6}$$

where m is the dimension of the input vector u. □

Moreover, the system (3.1) is *input observable* if B is monic and the unobservable subspace of (C, A) does not intersect the image of B. Moreover, if $u(t)$ is a scalar signal, i.e. $m = 1$, the input observability is equivalent to the left invertibility of the system, see [Mas86].
An FDI filter must be capable of isolating a fault occurrence from the others. To this end, the following concepts of invariant subspace and (C, A)-invariant subspace can be helpful.

Definition 8 ***Invariant subspaces*** *Given a subspace $\mathcal{W} \subset \mathcal{X} \subseteq \mathbb{R}^n$ and a linear map $A : \mathcal{X} \to \mathcal{X}$. The subspace $\mathcal{W}$ is said to be A-invariant if*

$$A\mathcal{W} \subseteq \mathcal{W} \tag{3.7}$$

□

Definition 9 (C, A)***-invariant subspace*** *Consider the system (3.1) and let $\mathcal{X}$, $\mathcal{Y}$ be, respectively, the state and the output subspace. A subspace $\mathcal{W}$ is said to be a (C, A)-invariant subspace if there exists a map $D : \mathcal{Y} \to \mathcal{X}$ such that*

$$(A + DC)\mathcal{W} \subseteq \mathcal{W} \tag{3.8}$$

or equivalently

$$A\left(\mathcal{W} \cap \mathcal{N}(C)\right) \subseteq \mathcal{W} \tag{3.9}$$

□

Moreover, the intersection of two (C, A)-invariant subspaces is a (C, A)-invariant subspace, while the sum is not. Thus, the set of all (C, A)-invariant containing a given subspace admits an infimal element. The next theorem introduces an algorithm in order to evaluate the least (C, A)-invariant subspace which contains a given subspace $\mathcal{L}$.

Theorem 2 *[BM69a] The least (C, A)-invariant containing a given subspace $\mathcal{L}$ is expressed by $\underline{\mathcal{W}}(\mathcal{L})$ and is defined by the recursive relationship*

$$W_k = \mathcal{L} + A(W_{k-1} \cap \mathcal{N}(C)) \qquad W_0 = \mathcal{L} \qquad k = 0, 1, \ldots \tag{3.10}$$

then there exists a $\bar{k} > 0$ such that for all $n > \bar{k}$

$$W_n = W_{\bar{k}} \tag{3.11}$$

then the least (C, A)-invariant subspace that contains $\mathcal{L}$ is given by

$$\underline{\mathcal{W}}(\mathcal{L}) = W_n \tag{3.12}$$

□

3.2 Affine dynamical system

Consider the following dynamical system

$$\begin{cases} \dot{x}(t) & = & A(\rho)x(t) + Bu(t) \\ y(t) & = & Cx(t) \end{cases} \tag{3.13}$$

with $A(\rho) = A_0 + \sum_{i=1}^{d} A_i\rho_i,\ A_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots d,\ B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Where $\rho = \rho(t)$ is a continuous bounded vector function of time

$$\rho : [0,\, \infty) \rightarrow \mathbb{R}^d \quad for\, all\, t \geq 0 \tag{3.14}$$

It is assumed that each entry of the vector $\rho = (\rho_1,\, \ldots,\, \rho_d)$ is bounded by extremal values $\underline{\rho}_j$ and $\overline{\rho}_j$

$$\rho_i \in \left[\underline{\rho}_j,\, \overline{\rho}_j\right] \tag{3.15}$$

The parameter vector ρ takes value in an hyper-rectangle, with the set Ω of its 2^d vertices defined by

$$\Omega \doteq \left\{(v_1, \ldots, v_d) : \quad v_j \in \{\underline{\rho}_j,\, \overline{\rho}_j\}\right\} \tag{3.16}$$

Thus, all the allowable trajectories of ρ are contained in the convex hull $co\Omega$ of the hyper-rectangle Ω.

Lemma 4 ***Quadratic Stability*** *The system (3.13) is said to be quadratically stable if there exists a positive definite matrix $P = P'$ such that*

$$A(\rho)'P + PA(\rho) \prec 0 \tag{3.17}$$

for all the trajectories $\rho \in co\Omega$. □

A Lyapunov function can be defined as $V(x) = x'Px$.

Lemma 5 ***Affine Quadratic Stability*** *The system (3.13) is said to be affine quadratically stable if there exists* $P_0, \cdots, P_d$ *symmetric matrices such that*

$$P(\rho) = P_0 + \rho_1 P_1 + \cdots + \rho_d P_d > 0 \tag{3.18}$$

$$A(\rho)' P(\rho) + P(\rho) A(\rho) + \frac{\partial P(\rho)}{\partial t} < 0 \tag{3.19}$$

for all the trajectories $\rho \in co\Omega$. □

An affine Lyapunov function can be defined as $V(x, \rho) = x' P(\rho) x$. As stated for the linear time-invariant system, also, for a system like (3.13) is possible to define the concept of invariance affine map.

Definition 10 *[SBS03] A subspace* $\mathcal{W}$ *is called parameter-varying invariant subspace for the family of linear maps* $A(\rho)$ *if*

$$A(\rho)\mathcal{W} \subset \mathcal{W} \tag{3.20}$$

for all $\rho \in co\Omega$. □

3.2.1 The Linear Parameter-Varying case

Suppose that the parameter $\rho(t)$ is measurable on-line, then (3.13) is said to be an Affine Linear Parameter Variable (ALPV) system. In order to design an FDI filter for such kind of systems, the concept of *uniform observability* is introduced.

Definition 11 *[SM67]* ***Uniform observability*** *A system like (3.13) is said to be uniformly observable for all trajectories of* $\rho(t)$ *if for any* t, t_0 *with* $t \geq t_0$, *any initial state* $x(t_0)$ *at* t_0 *can be determined from the knowledge of* $y(t)$ *and* $u(t)$. □

The next theorem introduces the terms needed such that there exists a stable observer for the system (3.13).

Theorem 3 *[WM71] Assume that the system (3.13) is uniformly observable for all trajectories of* $\rho(t)$. *Then there exists a bounded continuous matrix* $L(\rho(t))$ *such that* $A(\rho) - L(\rho)C$ *is exponentially stable.* □

Consider the following

Lemma 6 *If there exist matrices L_0, $L_1, \ldots,$ L_d*

$$L(\rho) = L_0 + \sum_{i=1}^{d} L_i \rho_i \tag{3.21}$$

and a positive definite matrix $P = P'$, such that the following LMI is feasible

$$A'_{cl}(\rho)P + PA_{cl(\rho)} \prec 0 \qquad \forall \rho \in co\Omega \tag{3.22}$$

with

$$A_{cl}(\rho) \doteq A(\rho) - L(\rho)C \tag{3.23}$$

Then, the system (3.13) is uniformly observable. □

See [AB95], [BP94], [SM67], [WM71] and [Szi92] for more details. Moreover in order to design well-posed FDI filters, it is necessary to determine the condition under which a fault is observable from the output signal $y(t)$. To this end the following definition is introduced

Definition 12 ***Input Observability*** *Consider (3.13), the input $u(t)$ is said to be observable if*

$$y(t) = 0 \quad \text{for } t \geq 0 \Rightarrow u(t) = 0 \quad \text{for } t > 0 \quad \forall \rho \in co\Omega \tag{3.24}$$

provided that $x(0) = 0$. □

In a similar way, one can introduce the extension of the concept of (C, A)-invariant subspace as

Definition 13 *[BBS03]* $(C, A(\rho))$***-invariant***
A subspace $\mathcal{W}$ is called $(C, A(\rho))$-invariant if there exists a map $L(\rho)$ such that

$$(A(\rho) + L(\rho)C)\mathcal{W} \subseteq \mathcal{W} \tag{3.25}$$

or equivalently

$$A(\rho)(\mathcal{W} \cap \mathcal{N}(C)) \subseteq \mathcal{W} \tag{3.26}$$

for all $\rho \in co\Omega$. □

The intersection of two $(C, A(\rho))$-invariant subspaces is an $(C, A(\rho))$-invariant subspace. Thus, the set of all $(C, A(\rho))$-invariant containing a given subspace admits an infimal element. The next theorem introduces an algorithm in order to evaluate the least $(C, A(\rho))$-invariant subspace which contains a given subspace $\mathcal{L}$.

Theorem 4 *[BBS03] The least $(C, A(\rho))$-invariant containing a given subspace $\mathcal{L}$ is expressed by $\underline{\mathcal{W}}_\rho(\mathcal{L})$ and is defined by the recursive relationship*

$$W_k = \mathcal{L} + \sum_{i=0}^{d} A_i(W_{k-1} \cap \mathcal{N}(C)) \qquad W_0 = \mathcal{L} \qquad k = 0, 1, \ldots \tag{3.27}$$

then there exists a $\bar{k} > 0$ such that for all $n > \bar{k}$

$$W_n = W_{\bar{k}} \tag{3.28}$$

then the least $(C, A(\rho))$-invariant subspace that contains $\mathcal{L}$ is given by

$$\underline{\mathcal{W}}_\rho(\mathcal{L}) = W_n \tag{3.29}$$

□

3.2.2 The Uncertainty case

Consider the case of $\rho(t)$ bounded vector function of time, but unknown. In such case the system (3.13) is subject to affine state uncertainties. The following lemmas introduce the terms such that for the uncertainty system (3.13) there exist a stable observer and a stabilizing constant state feedback regulator.

Lemma 7 *If there exist a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a feedback matrix $D \in \mathbb{R}^{n\times m}$ such that the following LMI is satisfied*

$$Q(A(\rho) - BD)' + (A(\rho) - BD)Q \prec 0 \quad \forall \rho \in co\Omega \tag{3.30}$$

then the pair $(A(\rho), B)$ is said to be uniformly stabilizable. Moreover, because the polytopic uncertainty is affine in the parameter ρ (3.30) is equal to

$$Q(A(v) - BD)' + (A(v) - BD)Q \prec 0 \quad \forall v \in \Omega \tag{3.31}$$

Lemma 8 *If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a matrix $L \in \mathbb{R}^{p\times n}$ such that the following LMI is satisfied*

$$(A(\rho) - LC)'P + P(A(\rho) - LC) \prec 0 \quad \forall \rho \in co\Omega \tag{3.32}$$

then the pair $(C, A(\rho))$ is said to be uniformly observable. Moreover, because the polytopic uncertainty is affine in the parameter ρ (3.32) is equal to

$$(A(v) - LC)'P + P(A(v) - LC) \prec 0 \quad \forall v \in \Omega \tag{3.33}$$

Definition 14 *[BBS02] A subspace $\mathcal{W}$ is a generalized $(C, A(\rho))$-invariant subspace if and only if there exists a matrix L such that for all $\rho \in co\Omega$*

$$(A(\rho) + LC)\mathcal{W} \subset \mathcal{W} \tag{3.34}$$

□

The intersection of two generalized $(C, A(\rho))$-invariant subspaces is a generalized $(C, A(\rho))$-invariant subspace. Thus, the set of all generalized $(C, A(\rho))$-invariant containing a given subspace admits an infimal element. The next theorem introduces an algorithm in order to evaluate the least generalized $(C, A(\rho))$-invariant subspace which contains a given subspace $\mathcal{L}$.

Theorem 5 *[BBS02] The least generalized $(C, A(\rho))$-invariant containing a given subspace $\mathcal{L}$ is expressed by $\underline{\mathcal{W}}_\rho(\mathcal{L})$ and is defined by the recursive relationship*

$$W_k = \mathcal{L} + A_0 \left(W_{k-1} \cap \mathcal{N}(C)\right) + \sum_{i=1}^{d} A_i W_{k-1} \qquad W_0 = \mathcal{L} \qquad k = 0, 1, \ldots \tag{3.35}$$

then there exists a $\bar{k} > 0$ such that for all $n > \bar{k}$

$$W_n = W_{\bar{k}} \tag{3.36}$$

then the least $(C, A(\rho))$-invariant subspace that contains $\mathcal{L}$ is given by

$$\underline{\mathcal{W}}_\rho(\mathcal{L}) = W_n \tag{3.37}$$

□

3.3 Conclusions

This chapter has introduced some geometric properties for LTI and ALPV systems. These notions will be helpful in the design of well-defined FDI filters.

Part II

FDI with a Sensitivity Constraint

This part explains the main results obtained in the research activity. A new fault detection and isolation algorithm is proposed. It is based on the design of a residual generator filter with a prescribed fault sensitivity while guaranteeing the best disturbance attenuation, even in the presence of multiple simultaneous faults. With fault sensitivity it is meant the extent of relationship between a fault and the residual. Thus, the objective is to enhance the fault detection capability, in such a way that the designed filters are able of detecting and isolating smaller fault signals. The algorithm is derived from solving a family of LMI optimization problems. Chapter 4 introduces the concept of fault detection and isolation residual generator design (FDIRG) problem on linear time invariant (LTI) systems subject to input disturbances and measurement noises. Initially, the problem of disturbance attenuation is addressed through an $\mathcal{H}_\infty$ approach. Section 4.3 introduces the basic idea of designing residual generators with a predetermined fault sensitivity level, which is carried out through an eigen-structure assignment. Hence, the FDIRG problem is solved through a linear matrix inequality (LMI) optimization problem. The major contribution is that if suitable assumptions are satisfied then the residual generator designing problem has always solution. In order to extend the results obtained on LTI systems, Chapter 5 explains the FDIRG problem on model uncertainty LTI systems. Chapter 6 explains the FDIRG problem on linear parametric variable (LPV) systems. Furthermore, the problem of enhancing the fault sensitivity capability of the filters is carried out.

Chapter 4

The LTI case

This chapter introduces a new algorithm for designing FDI residual generators for linear time invariant (LTI) systems subject to disturbances and measurement noise. The residual generator design method leads to the unknown-input observer approach. In particular the problem of enhancing the fault detection capability of the filter is addressed through an eigen-structure assignment. While the disturbances attenuation is solved through an optimal $\mathcal{H}_\infty$ filtering problem. With this geometric constraint it is possible to map the fault injection into a predetermined fixed residual subspace. Some assumptions are taken into account in order to make the FDIRG problem well-posed. Moreover, a ratio between the zero frequency gain of the fault to residual map and the $\mathcal{H}_\infty$ gain from the disturbances to residual map is defined as a performance criterion. Hence, the minimization of this ratio leads to a residual generator with the maximum fault and the minimum disturbance transmissions. In conclusion, the residual generator is obtained as the solution of a family of LMI optimization problems. Other approaches to this problem can be found in [CS02] and [EB02]. In [CS02], the authors face the problem via optimization techniques in such a way that the output error space is split in several subspaces and for each subspace the transmission from one specific fault is maximized while the transmission from other faults and noises is minimized. In [EB02], differently, a scaled $\mathcal{H}_\infty$ filtering problem is formulated so as to maintain a sufficient sensitivity to failure modes. This chapter is organized as follows. Section 4.1 explains the plant dynamical model and the filter model; moreover some assumptions will be introduced in order to make the problem well-posed. Section 4.2 explains the disturbance and nuisance attenuation problem. Then the optimization problem is introduced. Section 4.3 takes into account the fault sensitivity optimization and the attenuation problems. Modification are derived so as to obtain a new optimization problem that is capable of designing an FDI filter with enhanced detection and isolation properties. Section 4.4 explains the necessity of a projection matrix. Section 4.5 introduces the sensor fault case. Section 4.6 illustrates how to set a constraint in order that the filter's poles belong to a predetermined region of the complex plane. Section 4.7 explains a practical FDI filter design example, while Section 4.8 ends the chapter with some conclusions.

4.1 The dynamical model

Consider the following dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t)+Bu(t)+B_w w(t)+\sum_{i=1}^{\bar{r}} F_i f_i(t) \\ y(t) = Cx(t)+D_n n(t) \end{cases} \tag{4.1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}, B_w \in \mathbb{R}^{n\times \bar{w}}, D_n \in \mathbb{R}^{p\times \bar{n}}$, $C \in \mathbb{R}^{p\times n}$ and $F_i \in \mathbb{R}^n$ for $i = 1, \ldots, \bar{r}$. In (4.1), $x \in \mathbb{R}^n$ indicates the state, $u \in \mathbb{R}^m$ the input, $y \in \mathbb{R}^p$ the sensor output, $f_i \in \mathbb{R}$ the i-th failure signal, $w(t) \in \mathbb{R}^{\bar{w}}$ the plant disturbance, and $n(t) \in \mathbb{R}^{\bar{n}}$ the measurement noise. Notice that all the vectors n, w, f_i represent unknown input signals. It is worth pointing out, that all the signals are assumed to belong to the $\mathcal{L}_2$ functional space. Moreover, this assumption is not restrictive, because all the results that are valid for a signal that belongs to the $\mathcal{L}_2$ space can be extended to a signal that belongs to the $\mathcal{L}_{2T}$ space. Notice that, systems like (4.1) can be used in the filter design even in the case of sensor faults, because w.l.o.g. a sensor fault can be transformed in a pseudo-actuator fault, (see Section 1.2). The problem is to detect and isolate each fault occurrence f_i, i.e. to determine from the knowledge of the input $(u(t))$ and the output $(y(t))$ signals which are the fault signals $f_i(t)$ that are different from zero. Moreover, the filter should be capable of isolating each fault signal from the others in order to disclose the start time of each fault occurrence. The FDI filter structure that will be used is depicted in Fig. 4.1: each residual generator is capable of detecting and isolating one fault occurrence, in such a way one requires a number of residual generators equal to the number of the fault signals to be detected. For the sake of notational simplicity, in what follows

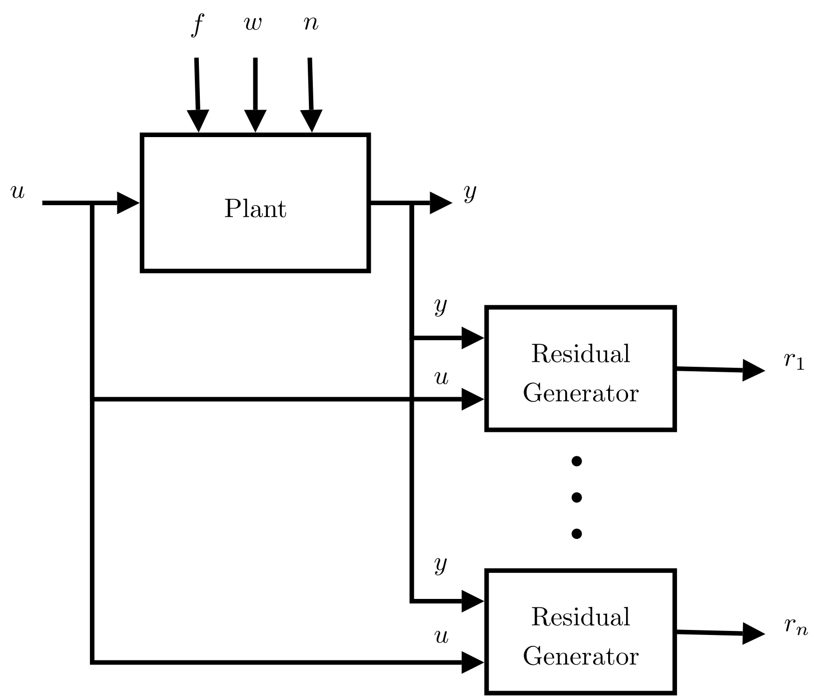


Figure 4.1. The residual generator

the case of detection and isolation of the i-th fault signal will be explained. Let the i-th fault signal to be detected and isolated, then the FDI problem amounts on designing a filter that has the following properties:

P-4.1. Whenever $f_i(t) \equiv 0$, the linear map from the nuisances (n, w, f_j with $j \neq i$) to the residual (r) has predetermined $\mathcal{H}_\infty$-gains.

P-4.2. Whenever $f_i(t) \neq 0$, the zero frequency gain from the fault ($f_i(t)$) to the residual ($r(t)$) has a predetermined minimal euclidean norm, irrespective of n, w, f_j with $j \neq i$.

In order to simplify the notation (4.1) can be rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t)+Bu(t)+B_w w(t)+F_i f_i(t)+\hat{F}_i \hat{f}_i(t) \\ y(t) = Cx(t)+D_n n(t) \end{cases} \tag{4.2}$$

where

$$\hat{F}_i \doteq [F_1,\, \dots,\, F_{i-1},\, F_{i+1},\, \dots,\, F_{\bar{r}}] \in \mathbb{R}^{n\times\bar{r}} \tag{4.3}$$

and

$$\hat{f}_i(t) \doteq [f_1(t),\, \dots,\, f_{i-1}, f_{i+1}(t),\, \dots,\, f_{\bar{r}}(t)]' \in \mathbb{R}^{\bar{r}} \tag{4.4}$$

with $\hat{f}_i(t)$ representing the nuisance signal. In order to design a residual generator capable of isolating the i-th fault occurrence, consider the following observer

$$\begin{cases} \dot{\hat{x}}(t) &= (A - LC)\hat{x}(t) + Bu(t) + Ly(t) \\ \hat{y}(t) &= C\hat{x}(t) \end{cases} \tag{4.5}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be designed and $\hat{x} \in \mathbb{R}^n$ and $\hat{y} \in \mathbb{R}^{p\times n}$ the state and the output estimates, respectively. Define the output estimation error as

$$\tilde{y} \doteq y - \hat{y} \tag{4.6}$$

and the residual as follows

$$r \doteq H\tilde{y} \tag{4.7}$$

where $H \in \mathbb{R}^{p\times p}$ is a projection matrix, whose role will be explained later. Then, L and H should be designed in such a way that

$$\begin{cases} r \approx 0 & f_i(t) \equiv 0 \\ r \neq 0 & f_i(t) \neq 0 \end{cases} \tag{4.8}$$

irrespective of $\hat{f}_i(t)$, $w(t)$ and $n(t)$. In what follows with *FDI problem* it will be meant the problem of finding matrices L and H such that Properties P-4.1 and P-4.2 are satisfied. The following assumptions are required for the solvability of the FDI problem:

A-4.1. The pair $(A,\, C)$ is observable.

A-4.2. All F_i, for $i = 1, \dots, \bar{r}$, are monic, viz. if $f_i(t) \neq 0$ then $F_i f_i(t) \neq 0$.

A-4.3. The $\text{rank}(C[F_1,\, \dots,\, F_{\bar{r}}]) = \bar{r}$.

A-4.4. The conditions $CA^kF_i \neq 0$, for $i = 1, \ldots, \bar{r}$, hold true for some $k = 0, \ldots, n-1$.

Assumption A-4.1 is necessary and sufficient for the existence of an asymptotically stable observer (4.5). Assumptions A-4.2 and A-4.4 are related to the *input observability* of the i-th fault. Assumption A-4.3 is referred to the *output separability* condition. It is advisable to introduce the state estimation error defined as

$$e \doteq x - \hat{x} \tag{4.9}$$

Consequently, it is possible to study the residual properties and to design the matrices L and H over the error dynamics system, defined as

$$\begin{cases} \dot{e}(t) &= (A - LC)e(t) + B_w w(t) - LD_n n(t) + F_i f_i(t) + \hat{F}_i \hat{f}_i(t) \\ \tilde{y}(t) &= Ce(t) + D_n n(t) \\ r(t) &= H\tilde{y}(t) \end{cases} \tag{4.10}$$

A necessary condition for the detectability and isolability of the i-th fault occurrence is that the signal f_i is input observable, i.e. $f_i(t) \neq 0 \Rightarrow r(t) \neq 0$. The following Lemma explains how the Assumptions A-4.2 and A-4.4 can guarantee that $f_i(t)$ is input observable from $r(t)$.

Lemma 9 *Let the Assumptions A-4.2 and A-4.4 hold true then there exists a matrix L such that the i-th fault is input observable from $\tilde{y}$.* □

Proof. Let the Assumption A-4.2 holds true and consider the unobservable subspace of $(C, A - LC)$ that is

$$\mathcal{U} = \mathcal{N}(C) \cap (A - LC)^{-1}\mathcal{N}(C) \cap \cdots \cap (A - LC)^{-n+1}\mathcal{N}(C) \tag{4.11}$$

Then to guarantee that the i-th fault is input observable the matrix F_i must be monic and the image of F_i does not intersect the unobservable subspace $\mathcal{U}$. Let the Assumption A-4.4 holds true, moreover suppose w.l.o.g. that $CF_i \neq 0$ then consider

$$\begin{aligned} \mathcal{U} \cap \mathcal{I}(F_i) \quad = \quad & (\mathcal{N}(C) \cap \mathcal{I}(F_i)) \cap ((A - LC)^{-1}\mathcal{N}(C) \cap \mathcal{I}(F_i)) \cap \\ & \cap \cdots \cap ((A - LC)^{-n+1}\mathcal{N}(C) \cap \mathcal{I}(F_i)) \end{aligned} \tag{4.12}$$

Notice that

$$\mathcal{N}(C) \cap \mathcal{I}(F_i) \equiv \emptyset \tag{4.13}$$

Moreover, it can be seen that $\mathcal{I}(F_i)$ is the smallest $(C, A-LC)$-invariant subspace, then

$$\begin{aligned}&(A-LC)^{-1}\mathcal{N}(C)\cap\mathcal{I}(F_i) = (A-LC)^{-1}(\mathcal{N}(C)\cap(A-LC)\mathcal{I}(F_i)) \subseteq \\ &\subseteq (A-LC)^{-1}(\mathcal{N}(C)\cap\mathcal{I}(F_i)) \equiv \emptyset\end{aligned} \tag{4.14}$$

In conclusion

$$\mathcal{U}\cap\mathcal{I}(F_i) \equiv \emptyset \tag{4.15}$$

then the i-th fault signal is input observable. Notice that if $CF_i = 0$ there exists a $k > 0$ such that $CA^kF_i \neq 0$ in such case the $\mathcal{I}(F_i)$ above can be replaced by $\mathcal{I}(A^kF_i)$. ■

Lemma 10 *If Assumption A-4.3 is satisfied each fault f_i, $i = 1, \dots, \bar{r}$ is isolable.* □

Recall that with *isolable* it is meant the property of the fault to be detected even in presence of other fault signals, see Section 1.

Proof. By contrary, suppose that there exists a fault map F_j with $j \neq i$ such that

$$CF_j = \alpha CF_i \tag{4.16}$$

with $\alpha \neq 0$ a real number. In this case the two faults have the same direction in the output subspace, In such a way that it is not possible to isolate the two faults. Then if Assumption A-4.3 is satisfied each fault can be mapped in a different output subspace, then there exists a projection matrix H that projects $\tilde{y}$ on the predefined output subspace. Notice that if $CF_i \equiv 0$, F_i is replaced by A^kF_i in the Assumption A-4.3, where k is such that the assumption A-4.4 holds true. ■

Proposition 1 *If assumptions A-4.2, A-4.3, A-4.4 hold true then all the r distinct fault signals $f_i(t)$ are input observable. Moreover, all $f_i(t)$ can be isolated by a suitable choice of the matrices L and H.* □

Proof. From Lemma 9 and 10 ■

4.2 The attenuation problem

In this section the disturbance attenuation problem is addressed through $\mathcal{H}_\infty$ design methods and solved via the solution of an LMI optimization problem. In order to satisfy the property P-4.1 it is advisable to impose the following constraint over $r(t)$

$$\|r\|_2^2 \leq \gamma^2\|\hat{f}_i\|_2^2 + \gamma_w^2\|w\|_2^2 + \gamma_n^2\|n\|_2^2 \tag{4.17}$$

Then, the best attenuation bound can be found through the solution of the following minimization problem

$$\begin{array}{rl} \min\limits_{L} & \mathbf{a}\gamma^2 + \mathbf{b}\gamma_w^2 + \mathbf{c}\gamma_n^2 \\ \textit{subject to} & \begin{cases} \|r\|_2^2 \leq \gamma^2\|\hat{f}_i\|_2^2 + \gamma_w^2\|w\|_2^2 + \gamma_n^2\|n\|_2^2 \\ (A - LC) \ \textit{asymptotically stable} \end{cases} \end{array} \tag{4.18}$$

with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ positive real weighting constants. In order to rewrite (4.18) as an LMI optimization problem, the following supply function is defined

$$s(r, [\hat{f}_i \quad w' \quad n']) = \begin{bmatrix} \hat{f}_i' & w' & n' \end{bmatrix} \begin{bmatrix} \gamma^2 & 0 & 0 \\ 0 & \gamma_w^2 I & 0 \\ 0 & 0 & \gamma_n^2 I \end{bmatrix} \begin{bmatrix} \hat{f}_i \\ w \\ n \end{bmatrix} - r'r \tag{4.19}$$

then from the Bounded Real Lemma the system (4.10) with (4.19) is dissipative if and only if the following LMI's are feasible

$$\begin{cases} \begin{bmatrix} A'P+PA-C'K'-KC+(HC)'HC & P\hat{F}_i & PB_w & -KD_n+C'H'HD_n \\ * & -\gamma^2 & 0 & 0 \\ * & 0 & -\gamma_w^2 I & 0 \\ * & 0 & 0 & (HD_n)'HD_n-\gamma_n^2 I \end{bmatrix} \prec 0 \\ P = P' \succ 0 \end{cases} \tag{4.20}$$

with

$$K \doteq PL \tag{4.21}$$

As explained in Section 2.1 the LMI numerical solvers are capable of solving an optimization problem which is linear in the variables and with a linear cost functional. To this end, since the constraint (4.20) is nonlinear in the variables γ, γ_w and γ_n it is necessary to introduce the following auxiliary variables

$$\bar{\gamma} = \gamma^2 \qquad \bar{\gamma}_w = \gamma_w^2 \qquad \bar{\gamma}_n = \gamma_n^2 \tag{4.22}$$

Then, (4.20) with this change of variables becomes

$$\begin{cases} \begin{bmatrix} A'P+PA-C'K'-KC+(HC)'HC & P\hat{F}_i & PB_w & -KD_n+C'H'HD_n \\ * & -\bar{\gamma} & 0 & 0 \\ * & 0 & -\bar{\gamma}_w I & 0 \\ * & 0 & 0 & (HD_n)'HD_n-\bar{\gamma}_n I \end{bmatrix} \prec 0 \\ P = P' \succ 0 \\ \bar{\gamma} > 0,\ \bar{\gamma}_w > 0,\ \bar{\gamma}_n > 0 \end{cases} \tag{4.23}$$

where $P \in \mathbb{R}^{n\times n}$ a positive definite matrix, $K \in \mathbb{R}^{n\times p}$, $\bar{\gamma} \in \mathbb{R}$, $\bar{\gamma}_n \in \mathbb{R}$ and $\bar{\gamma}_w \in \mathbb{R}$ are auxiliary variables. In conclusion (4.18) can be rewritten as

$$\begin{array}{rl} \min\limits_{P,K} & \mathbf{a}\bar{\gamma} + \mathbf{b}\bar{\gamma}_w + \mathbf{c}\bar{\gamma}_n \\ subject\ to & (4.23) \end{array} \tag{4.24}$$

The observer gain L can be obtained through the solution of (4.24) as follows

$$L = P^{-1}K \tag{4.25}$$

4.3 The sensitivity constraint

This section introduces the problem of enhancing the fault sensitivity. As a measure of the fault sensitivity the zero frequency gain from the fault to the residual together with an eigen-structure assignment will be considered. This constraint is useful to design a residual that converges in a predefined subspace. In order to explain how the sensitivity constraint can be introduced in the disturbance attenuation problem consider the error dynamics (4.10) relative to the only fault signal $f_i(t)$

$$\begin{cases} \dot{e}(t) &=& (A-LC)e(t) + F_i f_i(t) \\ \tilde{y}(t) &=& Ce(t) \\ r(t) &=& He(t) \end{cases} \tag{4.26}$$

then $\tilde{y}(t)$ can be written as

$$\tilde{y}(t) = Ce^{(A-LC)t}e(0) + C\int_{t_f}^{t} e^{(A-LC)(t-\tau)}F_i f_i(\tau)d\tau \tag{4.27}$$

where t_f indicates the start time of the i-th fault occurrence. The problem is to choose an appropriate matrix L such that $\tilde{y}(t)$ converges to a predefined subspace under the effect of the fault $f_i(t)$. To this end, the matrix L can be chosen in such a way to force F_i to be an eigenvector of $A - LC$,

$$(A - LC)F_i = \lambda F_i \tag{4.28}$$

for some real negative λ, if $CF_i \neq 0$. Then (4.27) can be rewritten as

$$\tilde{y}(t) = Ce^{(A-LC)t}e(0) + C\int_{t_f}^{t} e^{\lambda(t-\tau)}F_i f_i(\tau)d\tau \tag{4.29}$$

Therefore, as $t \longrightarrow \infty$ the $\tilde{y}(t)$ will converge in the subspace generated by the columns of CF_i. Moreover, with the constraint (4.28) it is also possible to set the zero frequency gain. The zero frequency gain from f_i to $\tilde{y}$ is

$$-\frac{1}{\lambda}CF_i \tag{4.30}$$

In the case of $CF_i = 0$, from the Assumption A-4.4 there exists a positive real k such that $CA^kF_i \neq 0$. Then, a more general constraint can be introduced

$$(A - LC)A^kF_i = \lambda A^kF_i \tag{4.31}$$

for some $k = 0, \dots, n-1$ and for some real negative λ. Consequently, (λ, A^kF_i) is forced to be an eigenvalue-eigenvector pair for $(A-LC)$. Then, (4.27) becomes

$$\tilde{y}_i(t) = Ce^{(A-LC)t}e(0) + CA^k\int_{t_f}^{t} e^{\lambda(t-\tau)}F_i f_i(\tau)d\tau \tag{4.32}$$

Hence, the residual lays in the subspace generated by the columns of CA^kF_i. Therefore, the zero frequency gain from fault f_i to residual r is defined as

$$G_i \doteq -\frac{1}{\lambda}HCA^kF_i \tag{4.33}$$

The fault sensitivity level is defined as the minimum singular value of G_i

$$\underline{\sigma}(G_i) = \underline{\sigma}(-\frac{1}{\lambda}HCA^kF_i) = \frac{1}{|\lambda|}\|HCA^kF_i\| \tag{4.34}$$

Notice that $\|HCA^kF_i\|$ is the euclidean norm of the vector $HCA^kF_i \in \mathbb{R}^p$. Moreover $\underline{\sigma}(G_i)$ is a function of λ.

Remark 1 *Notice that with (4.31) the parameter λ becomes a mode of the system (4.10), then if it is chosen too small the system will be slower.* □

The aim of the FDI problem is to design a residual generator filter with the smallest disturbance attenuation levels (γ, γ_w, γ_n) and the largest fault sensitivity level ($\underline{\sigma}(G_i)$). It is advisable to introduce a performance measure for the residual generator filter. A possible choice is to use the following ratio

$$\mu \doteq \frac{\mathbf{a}\gamma^2 + \mathbf{b}\gamma_w^2 + \mathbf{c}\gamma_n^2}{\underline{\sigma}(G_i)} = \frac{\lambda^2(\mathbf{a}\gamma^2 + \mathbf{b}\gamma_w^2 + \mathbf{c}\gamma_n^2)}{\|HCA^kF_i\|^2} \tag{4.35}$$

The smaller μ is, the better is the residual generator filter performance. Then, the disturbance attenuation problem can be extended in order to consider the enhancement of the fault sensitivity. Hence, (4.18) with the constraint (4.28) and the new cost functional (4.35) becomes

$$\begin{aligned} &\min_{L} \quad \lambda^2(\bar{\mathbf{a}}\gamma^2 + \bar{\mathbf{b}}\gamma_w^2 + \bar{\mathbf{c}}\gamma_n^2) \\ &subject\,to \quad \begin{cases} \|r_i\|_2^2 \leq \gamma^2\|\hat{f}_i\|_2^2 + \gamma_w^2\|w\|_2^2 + \gamma_n^2\|n\|_2^2 \\ (A - LC)A^kF_i = \lambda A^kF_i \\ (A - LC)\ asymptotically\ stable \end{cases} \end{aligned} \tag{4.36}$$

with $\bar{\mathbf{a}} = \frac{\mathbf{a}}{\|HCA^kF_i\|^2}$, $\bar{\mathbf{b}} = \frac{\mathbf{b}}{\|HCA^kF_i\|^2}$, $\bar{\mathbf{c}} = \frac{\mathbf{c}}{\|HCA^kF_i\|^2}$. In order to solve (4.36), the equality constraint (4.31) is considered. It is possible to make (λ, A^kF_i) an eigenvalue-eigenvector pair for the matrix $A - LC$, viz.

$$LCA^kF_i = (A - \lambda I)A^kF_i \tag{4.37}$$

with the extra condition that (λ, A^kF_i) is not an eigenvalue-eigenvector pair for the matrix A. The following lemma introduces the terms for the solvability of (4.37).

Lemma 11 *[PR94] If R, S and N are matrices of dimension $n \times m$, $m \times r$ and $n \times r$, respectively, where $n \geq m \geq r$ and $rank(S) = r$, then the general solution of $RS = N$, is given by*

$$R = NS^+ + E(I - SS^+) \tag{4.38}$$

where $E \in \mathbb{R}^{n\times m}$ is a free matrix and $S^+ \doteq (S'S)^{-1}S'$ is the left pseudo-inverse of S, while $E(I - SS^+)$ represents the freedom left in R after satisfying $RS = N$.
□

For the sake of notational simplicity, it is advisable to introduce the following matrix

$$A_{i,\lambda} \doteq (A - \lambda I)A^kF_i \tag{4.39}$$

It is easy to see that, all gains L that solve (4.37) can be parameterized in terms of a new free gain $\bar{L}$ as follows

$$L = A_{i,\lambda}(CA^kF_i)^+ + \bar{L}\left(I - CA^kF_i(CA^kF_i)^+\right) \tag{4.40}$$

where $(CA^kF_i)^+$ is the left pseudo inverse of CA^kF_i

$$(CA^kF_i)^+ \doteq \left[(CA^kF_i)'CA^kF_i\right]^{-1}(CA^kF_i)' \tag{4.41}$$

Notice that (4.41) always exists because CA^kF_i has a full column rank, for some k. Obviously, it is chosen $\lambda < 0$ for stability reasons. Taking into account the constraint (4.40) it is possible to rewrite (4.23) as follows

$$\begin{cases} \begin{bmatrix} \bar{A}'P+P\bar{A}-\bar{C}'K'-K\bar{C}+(HC)'HC & P\hat{F}_i & PB_w & -KD_n+C'H'HD_n \\ * & -\bar{\gamma} & 0 & 0 \\ * & 0 & -\bar{\gamma}_wI & 0 \\ * & 0 & 0 & (HD_n)'HD_n-\bar{\gamma}_nI \end{bmatrix} \prec 0 \\ P = P' \succ 0 \\ \bar{\gamma} > 0,\, \bar{\gamma}_w > 0,\, \bar{\gamma}_n > 0 \end{cases} \tag{4.42}$$

where

$$\begin{aligned} \bar{A} &\doteq A - A_{i,\lambda}(CA^kF_i)^+C && (4.43)\\ \bar{C} &\doteq (I - CA^kF_i(CA^kF_i)^+)C && (4.44)\\ K &\doteq P\bar{L} && (4.45) \end{aligned}$$

with $P \in \mathbb{R}^{n\times n}$ a positive definite matrix, $K \in \mathbb{R}^{n\times p}$ an auxiliary matrix variable and $\lambda < 0$. Moreover, a linear cost functional in λ, γ, γ_w and γ_n is needed. Consider the following inequality

$$\lambda^2(\bar{\mathbf{a}}\bar{\gamma} + \bar{\mathbf{b}}\bar{\gamma}_w + \bar{\mathbf{c}}\bar{\gamma}_n) \leq t \tag{4.46}$$

the optimization problem (4.36) can be rewritten as

$$\begin{array}{ll} \displaystyle\min_{P,K,\bar{\gamma},\bar{\gamma}_w,\bar{\gamma}_n} & t \\ \textit{subject to} & (4.42),\ (4.46) \end{array} \tag{4.47}$$

Notice that this problem involves constraints that are bilinear matrix inequalities. In order to solve this kind of problems via LMI optimization techniques it is possible to parameterize over λ the bilinear constraints. Therefore, (4.47) will be solved with λ as a fixed parameter chosen over a grid of values. The optimization problem (4.47) with λ fixed leads to an upper-bound of the solution of the same problem with λ a free parameter. Then, the sub-optimal solution is attained for the value $\bar{\lambda}$ over the grid at which μ is minimal. For solvability reasons, it is required also that the following assumption be satisfied

A-4.3.5. the pair $(\bar{A}, \bar{C})$ is detectable.

Otherwise a stable filter could not exist. Notice that the problem (4.47) is feasible if and only if the assumption A-4.3.5 holds true. If (4.47) is solvable, the matrix $\bar{L}$ is given by

$$\bar{L} = P^{-1}K \tag{4.48}$$

and

$$L = A_{i,\bar{\lambda}}(CA^kF_i)^+ + \bar{L}(I - CA^kF_i(CA^kF_i)^+) \tag{4.49}$$

Proposition 2 *Consider a dynamical system subject to simultaneous fault signals, like (4.1) and let the assumptions A-4.1-A-4.3.5 be satisfied. Then the problem (4.47) has always a feasible solution.*

4.4 The projection Matrix

In this section the geometric aspect of the eigenvalue-eigenvector assignment is considered. As shown in the previous section the sensitivity constraint can set the residual to converge in the subspace generated by the columns of CA^kF_i, where k is the least integer such that $CA^kF_i \neq 0$. Then, the projection matrix H can be chosen in such a way to project $\tilde{y}$ in this subspace. It is useful to recall the system (4.26)

$$\begin{cases} \dot{e}(t) &= (A - LC)e(t) + F_i f_i(t) \\ \tilde{y}(t) &= Ce(t) \\ r(t) &= He(t) \end{cases}$$

and the eigenvalue-eigenvector assignment (4.37)

$$LCA^kF_i = (A - \lambda I)A^kF_i$$

The reachability matrix $\mathbf{R}$ of system (4.26) is

$$\mathbf{R} = \begin{bmatrix} F_i & (A - LC)F_i & \cdots & (A - LC)^{n-1}F_i \end{bmatrix} \tag{4.50}$$

Taking into account the sensitivity constraint (4.37) the reachability matrix becomes

$$\mathbf{R} = \begin{bmatrix} F_i & \lambda A^k F_i & \cdots & \lambda^{n-k-1} A^k F_i \end{bmatrix} \tag{4.51}$$

Notice that $\mathbf{R}$ has at least rank one and, however, not higher than two whatever is the value of $\lambda \neq 0$. Then, a fault signal f_i will be mapped in the error space along the direction $[F_i \quad A^k F_i]$. The output space can be mapped by the subspace generated by the columns of

$$C\mathbf{R} = \begin{bmatrix} CF_i & \lambda CA^k F_i & \cdots & \lambda^{n-k-1} CA^k F_i \end{bmatrix} \tag{4.52}$$

If $CF_i \neq 0$ then $k = 0$ and the output subspace is defined by

$$\mathcal{I}(C\mathbf{R}) = \mathcal{I}(CF_i) \tag{4.53}$$

else if $CF_i = 0$ there exists some positive integer k such that $CA^k F_i \neq 0$ the output subspace is defined by

$$\mathcal{I}(C\mathbf{R}) = \mathcal{I}(CA^k F_i) \tag{4.54}$$

Then, in the generic case the fault signal f_i will be mapped in the output space along the direction $CA^k F_i$. In conclusion the residual generator is capable of mapping a fault signal f_i in a fixed and predefined subspace. In order to reduce the effects of other nuisance it is advisable to project the residual r over the output subspace (4.54). This can be done with the use of a projection matrix defined as follows

$$H = I - C\hat{F}_i (C\hat{F}_i)^+ \tag{4.55}$$

where if some columns j of $\hat{F}_i$ is such that $CF_j = 0$, it can be replaced with $A^k F_j$ with k the least positive real such that $CA^k F_j \neq 0$.

4.5 The sensor fault case

This section explains the application of the sensitivity constraint in the case where the system (4.2) is subject to a sensor fault. Recall that a sensor fault can be mapped as a pseudo-actuator fault. To this end consider the case when the i-th fault is a sensor fault.

$$y(t) = Cx(t) + E_i f_i(t) \tag{4.56}$$

with $E_i \in \mathbb{R}^p$ and $f_i(t)$ a scalar fault signal. As shown in Section 1.2 it is possible to map the sensor fault into a pseudo-actuator fault, if there exists a matrix $F_i \in \mathbb{R}^n$ such that

$$CF_i = E_i \tag{4.57}$$

then the sensor fault can be written as a pseudo-actuator fault

$$F_i^{pa} = [F_i \; AF_i] \tag{4.58}$$

The eigenvalue-eigenvector constraint becomes

$$(A - LC)F_i = \lambda F_i \tag{4.59}$$

$$(A - LC)AF_i = \lambda AF_i \tag{4.60}$$

Then it is possible to impose the eigenvalue-eigenvector constraint in the following way.

$$L = (A - \lambda I)F_i(CF_i)^+ M + (A - \lambda I)AF_i(CAF_i)^+ N + \bar{L}(I - CF_i^{pa}(CF_i^{pa})^+) \tag{4.61}$$

where M and N are matrices which satisfy the constraints

$$\begin{cases} MCF_i = CF_i \\ MCAF_i = 0 \end{cases} \quad \begin{cases} NCF_i = 0 \\ NCAF_i = CAF_i \end{cases} \tag{4.62}$$

Then $\bar{A}$ and $\bar{C}$ in (4.43)-(4.44) become

$$\bar{A} = A - \big((A - \lambda I)F_i(CF_i)^+ M + (A - \lambda I)F_i(CAF_i)^+ N\big)\, C \tag{4.63}$$

$$\bar{C} = (I - CF_i^{pa}(CF_i^{pa})^+)C \tag{4.64}$$

4.6 The pole placing

Notice that it is also possible to consider a pole constrained region for the error dynamics $A - LC$. Let $\mathcal{D}$ be a region of the complex plane, defined as

$$\mathcal{D} \doteq \{z \in \mathbb{C} : L + zM + \bar{z}M' < 0\} \tag{4.65}$$

where $L = L'$ and M are real matrices. Then we can add a pole constraint by simply adding the following LMI in (4.47)

$$L \otimes P + M \otimes (P\bar{A} - K\bar{C}) + M' \otimes (\bar{A}'P - \bar{C}'K') \preceq 0 \tag{4.66}$$

where $\otimes$ indicates the Kronecker product. See Chapter 2 and [CGA99] for a complete description of a complex plane region with (4.65).

4.7 Example

This section explains the FDI filter design method introduced in the previous sections, through a practical example. Consider the following MIMO system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + F_1 f1(t) + F_2 f_2(t) \\ y(t) &= Cx(t) \end{cases} \tag{4.67}$$

where

$$A = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.5 & 0.6 \\ 0 & -0.6 & -0.5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$F_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{4.68}$$

The system has a fault for each actuator, they are distinct and indicated as f_1 and f_2. It is easily to verify that the Assumption A-4.1-A-4.4 hold true. In order to detect and isolate the fault signals f_1 and f_2, a bank of two residual generator will be designed. In the following, the subscript $_1$ and $_2$ indicate the matrices of the residual generators related to the fault f_1 and f_2, respectively. Then, the projection matrices H_i, $i = 1$, 2, for each residual generator are

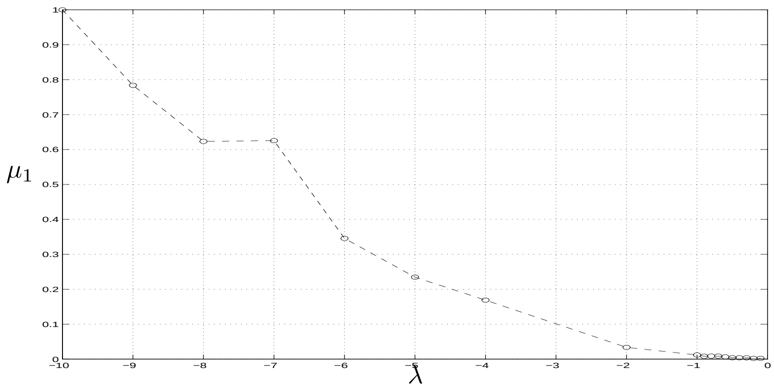
$$H_1 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix} \tag{4.69}$$

Define the performance ratio for each residual generator as

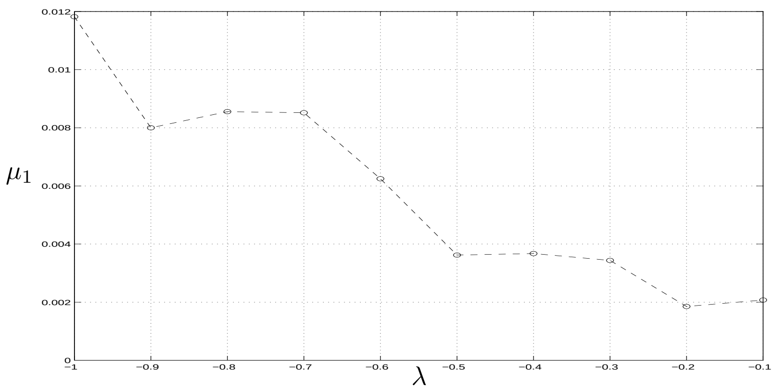
$$\mu_1 \doteq \frac{\gamma_1^2 \lambda_1^2}{\|H_1 C F_1\|^2} \tag{4.70}$$

$$\mu_2 \doteq \frac{\gamma_2^2 \lambda_2^2}{\|H_2 C F_2\|^2} \tag{4.71}$$

Fig. 4.2 and 4.3 illustrate a normalized plot of μ_1 and, respectively, μ_2 trend for a grid of λ values in $[-10\ -0.1]$. The frequency behavior can be helpful in



(a) $\lambda \in (-10\ -0.1)$



(b) $\lambda \in (-1\ -0.1)$

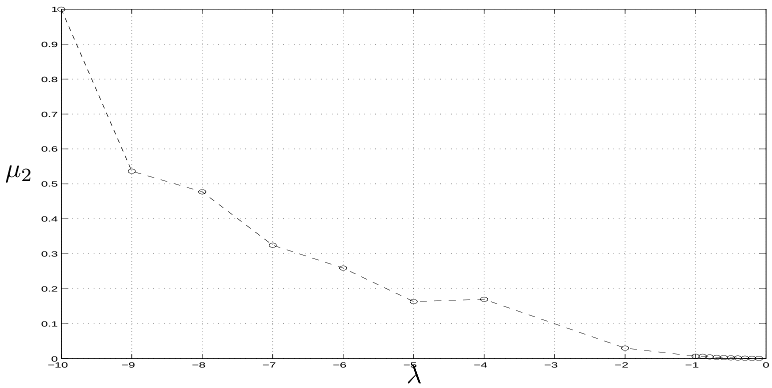
Figure 4.2. μ_1 vs. λ

understanding the benefits of the FDI filter design method. Then, the frequency response of the residual generator filters from faults to residual will be analyzed. To this end, the transfer function from the fault (f_1) to the residual (r_1) is defined by

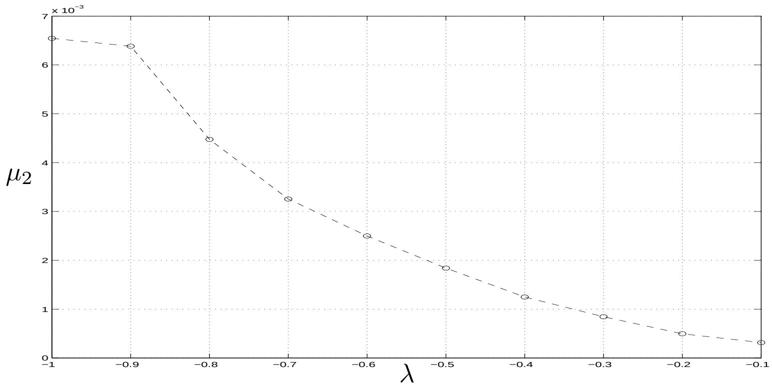
$$G_{f_1 r_1}(j\omega) \doteq H_1 C(j\omega - A + L_1 C)^{-1} F_1 \tag{4.72}$$

and the transfer function from the nuisance (f_2) to the residual (r_1) is

$$G_{f_2 r_1}(j\omega) \doteq H_1 C(j\omega - A + L_1 C)^{-1} F_2 \tag{4.73}$$



(a) $\lambda \in (-10\ -0.1)$



(b) $\lambda \in (-1\ -0.1)$

Figure 4.3. μ_2 vs. λ

In the same way, the transfer functions for the second residual generator are defined by

$$G_{f_1r_2}(j\omega) \doteq H_2C(j\omega - A + L_2C)^{-1}F_1 \quad (4.74)$$

$$G_{f_2r_2}(j\omega) \doteq H_2C(j\omega - A + L_2C)^{-1}F_2 \quad (4.75)$$

Where L_1 and L_2 are the observer matrices. For the sake of treatment simplicity, define the singular value $S(j\omega)$ by

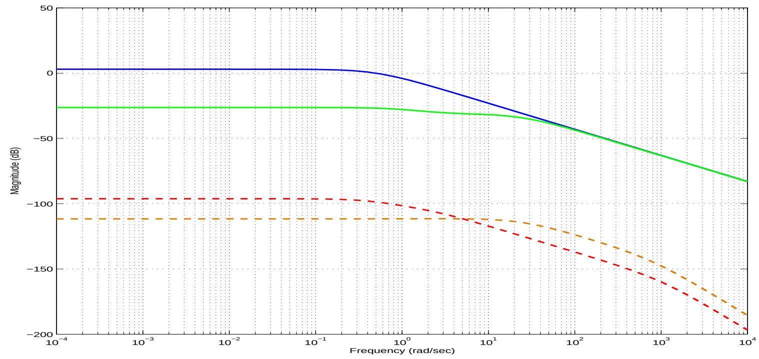
$$S(j\omega) \doteq G^*(j\omega)G(j\omega) \quad (4.76)$$

where $G^*(j\omega) = G'(-j\omega)$ with $G(j\omega)$ one of the predefined transfer functions in (4.72)-(4.75). In order to show the effects of the eigen-structure constraint, the singular values obtained by the solution of the attenuation problem (4.24) and by the solution of the attenuation problem with a sensitivity constraint (4.47) are compared. Fig. 4.4(a) illustrates the singular value plots related to the first

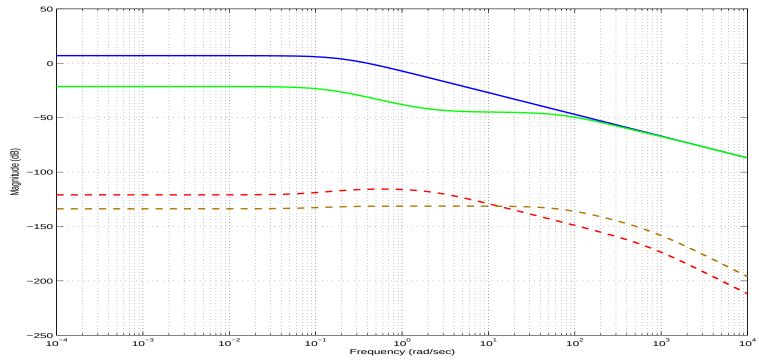
	Fig. 4.4(a)	Fig. 4.4(b)
LMI SC	$S_{f_1r_1}(j\omega)$	$S_{f_2r_2}(j\omega)$
	$S_{f_2r_1}(j\omega)$	$S_{f_1r_2}(j\omega)$
LMI	$S_{f_1r_1}(j\omega)$	$S_{f_2r_2}(j\omega)$
	$S_{f_2r_1}(j\omega)$	$S_{f_1r_2}(j\omega)$

Table 4.1. Legend for Fig. 4.4

residual generator. As it can be seen from Fig. 4.4 the ratios $\frac{S_{f_1r_1}(0)}{S_{f_2r_1}(0)}|_{dB}$ and $\frac{S_{f_2r_2}(0)}{S_{f_1r_2}(0)}|_{dB}$ are greater for the residual generators obtained as solution of the problem (4.47). Notice that the enhancements are also for higher frequencies and not only for the zero frequency. Fig. 4.5 illustrates the case where singular value plot are obtained from the residual generators without using the projection matrices H_i. Notice that the use of the projection matrix is useful to enhance the ratio $\frac{S_{f_ir_i}(0)}{S_{f_jr_i}(0)}|_{dB}$, $i = 1, 2$ and $j \neq i$. In particular the benefits of the projection matrix are visible at the higher frequencies.



(a) The first residual generator

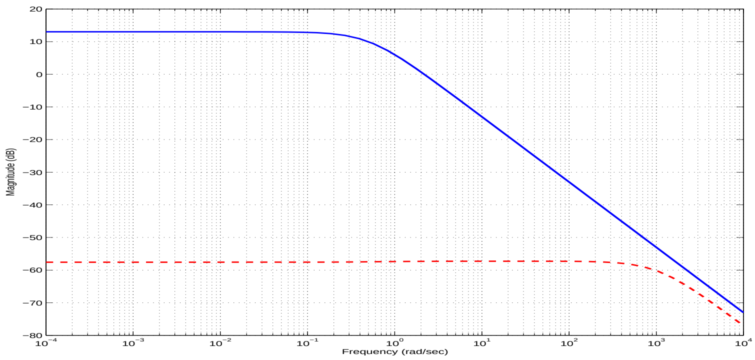


(b) The second residual generator

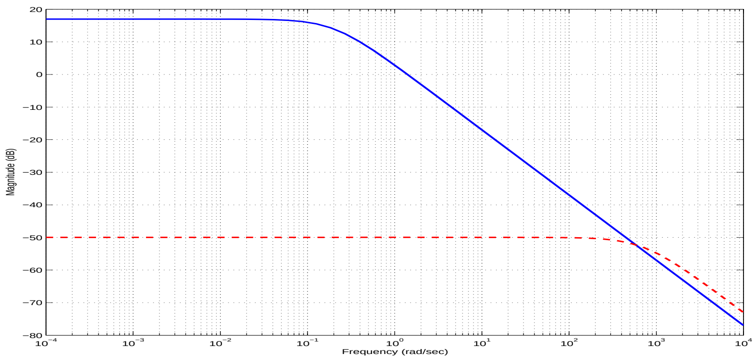
Figure 4.4. The singular value plot

	Fig. 4.5(a)	Fig. 4.5(b)
LMI SC	— (solid blue) $S_{f_1 r_1}(j\omega)$ - - - (dashed red) $S_{f_2 r_1}(j\omega)$	— (solid blue) $S_{f_2 r_2}(j\omega)$ - - - (dashed red) $S_{f_1 r_2}(j\omega)$

Table 4.2. Legend for Fig. 4.5



(a) The first residual generator without projection



(b) The second residual generator without projection

Figure 4.5. The singular value plot

4.8 Conclusions

This chapter has introduced a new design procedure to enhance the fault sensitivity in fault detection filters. The fault sensitivity is obtained through an eigen-structure assignment while the attenuation problem is solved through a $\mathcal{H}_\infty$ filtering design technique. Hence, an optimization problem is introduced in order to design the residual generator filter with the best fault transmission capability and the minimum $\mathcal{H}_\infty$ gain from disturbances to residual. The optimization problem introduced is a bilinear matrix inequality (BMI) optimization problem, which is a non-convex problem. In order to solve the problem with convex optimization methods a family of linear matrix inequality (LMI) optimization problem is used to obtain an upper-bound of the initial non-convex problem. In conclusion some assumptions are introduced in order to have a well-posed LMI optimization problem. Hence, if a linear system satisfies these assumptions a stable residual generator filter can be designed. A practical example shows the benefits introduced by the eigen-structure assignment in the overall filter performances, with respect to a residual generator designed taking into account the only disturbance attenuation problem.

Chapter 5

The LTI case with model uncertainties

This chapter introduces the design of residual generator filters for systems subject to model uncertainties. The design is carried out in order to enhance the fault detection capability of the filter. The filter design method leads to the unknown-input observer approach. In particular the problem of enhancing the fault detection capability of the filter is addressed through an eigen-structure assignment. While, the disturbances attenuation is solved through an optimal $\mathcal{H}_\infty$ filtering problem. Some assumptions are taken in order to make the FDIRG problem well-posed. Moreover, a ratio between the zero frequency gain of the fault to residual map and the $\mathcal{H}_\infty$ gain of the disturbances to residual map is defined as a performance criterion. Then, the minimization of this ratio leads to a residual generator with the maximum fault and the minimum disturbance transmissions to the residual signal. In conclusion, the residual generator is obtained as a solution of a family of LMI optimization problems. Other approaches to this problem can be found in [EBSK97]. In which, the authors solves the problem by applying a linear time-invariant equivalent representation to the original system. Hence, the residual generator filter is designed using an algebraic approach. This chapter is organized as follows. Section 5.1 explains the plant dynamical model and some assumptions are introduced in order to make the problem well-posed. Section 5.2 explains the design of a residual generator. Section 5.3 explains a practical FDI filter design example, while Section 5.4 ends the chapter with some conclusions.

5.1 The dynamical model

Consider the following representation of a system with state affine uncertainty:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + Bu(t) + \sum_{i=1}^{\bar{r}} F_i f_i(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{5.1}$$

where $\rho \in \mathbb{R}^d$ is an uncertainty vector and $A(\rho)$ is defined in an affine way by

$$A(\rho) \doteq A_0 + \sum_{j=1}^{d} \rho_j A_j \tag{5.2}$$

with $A_j \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $F_i \in \mathbb{R}^n$ and ρ_j the entries of the uncertainty vector, which are bounded with known extremal

$$\rho_j \in \left[\underline{\rho}_j,\, \bar{\rho}_j\right] \tag{5.3}$$

The uncertainty vector ρ takes values in an hyper-rectangle, with the set Ω of its 2^d vertices defined by

$$\Omega \doteq \left\{(v_1, \ldots, v_d) : \quad v_j \in \{\underline{\rho}_j,\, \bar{\rho}_j\}\right\} \tag{5.4}$$

while with $co\Omega$ is indicated the convex hull of Ω which contains all possible values of ρ. In (5.1), $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ indicate the state and, respectively, the input vector, while $f_i(t)$ for $i = 1, \ldots, \bar{r}$ represent $\bar{r}$ distinct faults signals. The fault detection and isolation of simultaneous fault signals is carried out trough the design of a bank of residual generator, each one capable of isolating a single fault occurrence. It is worth pointing out that all the signals lay in the $\mathcal{L}_2$ functional space. The next sections address the residual generator design in such a way that the properties hereafter are satisfied

P-5.1. Whenever $f_i(t) \equiv 0$, the linear map from the nuisances (f_j with $j \neq i$) to the residual (r) has a predetermined $\mathcal{H}_\infty$-gain γ, for all $\rho \in co\Omega$.

P-5.2. Whenever $f_i(t) \neq 0$, the zero frequency gain from the fault (f_i) to the residual (r) has a predetermined minimum euclidean norm, for all $\rho \in co\Omega$ and irrespective of $f_j(t) \neq 0$ with $j \neq i$.

For the sake of notational simplicity (5.1) can be rewritten as

$$\begin{cases} \dot{x}(t) &= A(\rho)x(t) + Bu(t) + F_i f_i(t) + \hat{F}_i \hat{f}_i(t) \\ y(t) &= Cx(t) \end{cases} \tag{5.5}$$

where

$$\hat{F}_i \doteq [F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_{\bar{r}}] \in \mathbb{R}^{n \times \bar{r}} \tag{5.6}$$

and

$$\hat{f}_i(t) \doteq [f_1(t), \ldots, f_{i-1}, f_{i+1}(t), \ldots, f_{\bar{r}}(t)]' \in \mathbb{R}^{\bar{r}} \tag{5.7}$$

where $\hat{f}_i(t)$ indicates the nuisance signal. The following assumptions are required in order to have the FDI problem well-posed:

A-5.1. The pair $(A(\rho), B)$ is uniformly stabilizable.

A-5.2. The pair $(A(\rho), C)$ is uniformly observable.

A-5.3. For $i = 1, \ldots, \bar{r}$, all F_i are monic, viz. if $f_i(t) \neq 0$ then $F_i f_i(t) \neq 0$.

A-5.4. The conditions $CF_i \neq 0$, for $i = 1, \ldots, \bar{r}$ hold true.

A-5.5. The $\text{rank}(C[F_1, \ldots, F_{\bar{r}}]) = \bar{r}$.

The Assumptions A-5.1 and A-5.2 are necessary and sufficient conditions for the existence of a stable residual generator filter. The Assumptions A-5.3-A-5.4 are related to the input observability property, i.e. the capability of a signle fault occurrence to be detected from the inputs and the outputs of the system. The Assumption A-5.5 is related to the output separability property.

5.2 The FDI Problem with a Sensitivity Constraint

The FDI problem consists of designing a residual generator filter capable of detecting and isolating the i-th fault occurrence irrespective of $u(t)$ and $f_j(t)$,

$j \neq i$, in order to be robust with respect to the state uncertainty. Consider the following observer

$$\begin{cases} \dot{\hat{x}}(t) & = & A_0\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \\ \hat{y}(t) & = & C\hat{x}(t) \end{cases} \tag{5.8}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain matrix, to be determined. In (5.8), $\hat{x} \in \mathbb{R}^n$ indicates the observer state, while $\hat{y} \in \mathbb{R}^p$ is the output estimation. The output estimation error is defined as

$$\tilde{y} \doteq y - \hat{y} \tag{5.9}$$

and the residual as

$$r \doteq H\tilde{y} \tag{5.10}$$

where $H \in \mathbb{R}^{p\times p}$ is a projection matrix, to be determined. The state error is defined as

$$e(t) \doteq x(t) - \hat{x}(t) \tag{5.11}$$

then, the residual generator dynamics is defined as follows

$$\begin{cases} \dot{e}(t) & = & (A_0 - LC)e(t) + (A(\rho) - A_0)x(t) + F_i f_i(t) + \hat{F}_i\hat{f}_i(t) \\ \tilde{y}(t) & = & Ce(t) \\ r(t) & = & H\tilde{y}(t) \end{cases} \tag{5.12}$$

For the designing of matrices L and H, it will be useful the introduction of a residual generator extended dynamics, with the extended state defined by

$$\mathcal{X}(t) \doteq \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{5.13}$$

Then, the residual generator extended dynamics becomes

$$\begin{cases} \dot{\mathcal{X}}(t) & = & \mathcal{A}(\rho)\mathcal{X}(t) + \mathcal{B}u(t) + \mathcal{F}_i f_i(t) + \hat{\mathcal{F}}_i\hat{f}_i(t) \\ \tilde{y}(t) & = & \mathcal{C}\mathcal{X}(t) \\ r(t) & = & H\mathcal{C}\mathcal{X}(t) \end{cases} \tag{5.14}$$

where

$$\begin{aligned} \mathcal{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ A(\rho) - A_0 & A_0 - LC \end{bmatrix} \quad \mathcal{C} = \begin{bmatrix} 0 & C \end{bmatrix} \\ \mathcal{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \mathcal{F}_i = \begin{bmatrix} F_i \\ F_i \end{bmatrix} \quad \hat{\mathcal{F}}_i = \begin{bmatrix} \hat{F}_i \\ \hat{F}_i \end{bmatrix} \end{aligned} \tag{5.15}$$

The identification of the i-th fault occurrence is related to its input observability. The next lemma introduces the terms needed by (5.14) such that f_i is input observable

Lemma 12 *Let the Assumptions A-5.1-A-5.4 be satisfied then there exists a matrix L such that $f_i(t)$ is input observable from $\tilde{y}(t)$* □

Proof. Consider

$$\begin{cases} \dot{\mathcal{X}}(t) &= \mathcal{A}(\rho)\mathcal{X}(t) + \mathcal{B}u(t) + \mathcal{F}_i f_i(t) \\ \tilde{y}(t) &= \mathcal{C}\mathcal{X}(t) \\ r(t) &= H\mathcal{C}\mathcal{X}(t) \end{cases} \tag{5.16}$$

From Assumptions A-5.1 and A-5.2 there exist a pair of matrices D and L such that $A(\rho) + BD$ and $A_0 - LC$ are quadratically stable. Define

$$A_{cl}(\rho) \doteq \begin{bmatrix} A(\rho) + BD & 0 \\ A(\rho) - A_0 & A_0 - LC \end{bmatrix} \tag{5.17}$$

Then the system

$$\begin{cases} \dot{\mathcal{X}}(t) &= \mathcal{A}_{cl}(\rho)\mathcal{X}(t) + \mathcal{F}_i f_i(t) \\ \tilde{y}(t) &= \mathcal{C}\mathcal{X}(t) \\ r(t) &= H\mathcal{C}\mathcal{X}(t) \end{cases} \tag{5.18}$$

is quadratically stable. For the sake of notational simplicity define

$$A_j^K \doteq A_j + A_0 + BD \tag{5.19}$$

$$A_j^0 \doteq A_j - A_0 \tag{5.20}$$

Let the Assumption A-5.4 holds true, then the smallest generalized $(C, \mathcal{A}(\rho))$-invariant subspace is equal to

$$\underline{\mathcal{W}}(\mathcal{F}_i) = \mathcal{I}\left(\begin{bmatrix} F_i & \sum_{n_1} A_{n_1}^K F_i & \sum_{n_2}\sum_{n_1} A_{n_2}^K A_{n_1}^K F_i & \cdots \\ F_i & \sum_{n_1} A_{n_1}^0 F_i & \sum_{n_2}\sum_{n_1} A_{n_2}^0 A_{n_1}^K F_i & \cdots \end{bmatrix}\right.$$
$$\left.\begin{matrix} \sum_{n_n}\sum_{n_{n-1}}\cdots\sum_{n_1} A_{n_n}^K A_{n_{n-1}}^K \cdots A_{n_1}^K F_i \\ \sum_{n_n}\sum_{n_{n-1}}\cdots\sum_{n_1} A_{n_n}^0 A_{n_{n-1}}^K \cdots A_{n_1}^K F_i \end{matrix}\ \right]\right) \tag{5.21}$$

with

$$\underline{\mathcal{W}}(\mathcal{F}_i) \nsubseteq \mathcal{N}(\mathcal{C}) \tag{5.22}$$

Suppose that $\mathcal{X}(0) \neq 0$ and $f_i(t) \neq 0$ then for the quadratic stability of (5.18) all trajectories converge to the subspace $\underline{\mathcal{W}}(\mathcal{F}_i)$, then $\tilde{y} \neq 0$ because this subspace is not contained in the unobservable subspace of $\mathcal{C}$. Suppose that $\mathcal{X}(0) = 0$ and $f_i(t) \neq 0$ then all the trajectories lay in $\underline{\mathcal{W}}(\mathcal{F}_i)$, then $\tilde{y} \neq 0$. Moreover, the output subspace is mapped by

$$\mathcal{C}\underline{\mathcal{W}}(\mathcal{F}_i) = \mathcal{I} \quad \Big(\Big[\; F_i \quad \sum_{n_1} A^0_{n_1} F_i \quad \sum_{n_2}\sum_{n_1} A^0_{n_2} A^K_{n_1} F_i \quad \cdots \\ \sum_{n_n}\sum_{n_{n-1}} \cdots \sum_{n_1} A^0_{n_n} A^K_{n_{n-1}} \cdots A^K_{n_1} F_i \Big]\Big) \tag{5.23}$$

■

In order to satisfy the property P-5.1 the matrix L will be designed such that (5.14) has an $\mathcal{H}_\infty$-gain γ, from the nuisance to the residual, guaranteed for all allowable values of the uncertainty vector ρ.

$$\|r\|_2^2 \leq \gamma^2 \|\hat{f}_i\|_2^2 \tag{5.24}$$

Then the preliminary nuisance attenuation problem can be addressed as the problem of finding L satisfying (5.24) and minimizing γ.

$$\begin{array}{ll} \min\limits_{L} & \gamma^2 \\ \textit{subject to} & \begin{cases} \|r\|_2^2 \leq \gamma^2 \|\hat{f}_i\|_2^2 \\ A_0 - LC \quad \textit{quadratically stable} \\ \forall \rho \in co\Omega \end{cases} \end{array} \tag{5.25}$$

To solve (5.25) a Lyapunov function is introduced

$$V(x) = x' P x \tag{5.26}$$

with $P \in \mathbb{R}^{n \times n}$ a symmetric positive definite matrix. For the sake of treatment simplicity from now on, it is assumed that $A(\rho)$ is quadratically stable. Then, (5.14) is quadratically stable with $\mathcal{H}_\infty$-gain γ if the next inequality is satisfied for all $\rho \in co\Omega$

$$\frac{dV(x)}{dt} + r'r - \gamma^2 \hat{f}_i' \hat{f}_i < 0 \tag{5.27}$$

A sufficient condition for (5.27) is that the next LMI constraints hold true

$$\begin{cases} \begin{bmatrix} \mathcal{A}(\rho)'P + P\mathcal{A}(\rho) & P\hat{\mathcal{F}}_i & (H\mathcal{C})' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ P = P' \succ 0 \end{cases} \tag{5.28}$$

for all $\rho \in co\Omega$. Notice that this introduces an infinite number of constraints, which can be reduced to a finite number taking into account the affine relation between the parameter ρ and (5.28). One has that if (5.28) is satisfied over the vertices $\rho \in \Omega$ then (5.27) holds true for all allowable values of ρ. Let the matrix P be defined as

$$P \doteq \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \succ 0 \tag{5.29}$$

with P_1, $P_2 \in \mathbb{R}^{n \times n}$ positive definite symmetric matrices. Taking into account (5.15) and (5.29) the LMI (5.28) can be rewritten as

$$\begin{bmatrix} A(\rho)'P_1 + P_1 A(\rho) & (A(\rho) - A_0)' P_2 & P_1 \hat{F}_i & 0 \\ * & A_0' P_2 + P_2 A_0 - C'K' - KC & P_2 \hat{F}_i & (HC)' \\ * & * & -\gamma I & 0 \\ * & * & 0 & -\gamma I \end{bmatrix} \preceq 0 \tag{5.30}$$

with K defined by

$$K \doteq P_2 L \tag{5.31}$$

The problem (5.25) can be rewritten as

$$\min_{P_1,\, P_2,\, K,\, \gamma} \quad t$$
$$\textit{subject to} \quad \begin{cases} (5.30) \\ \begin{bmatrix} t & \gamma \\ \gamma & 1 \end{bmatrix} \succ 0 \\ \gamma > 0 \end{cases} \tag{5.32}$$

where the nonlinear cost functional γ^2 is rewritten as a constraint in order to have a convex optimization problem with linear cost functional. Thanks to the Schur complement, the constraint

$$\gamma^2 \leq t \tag{5.33}$$

becomes

$$\begin{bmatrix} t & \gamma \\ \gamma & 1 \end{bmatrix} \succ 0 \tag{5.34}$$

In order to satisfy the property (P-5.2) consider $\tilde{y}(t)$

$$\tilde{y}(t) = \mathcal{C} e^{\mathcal{A}(\rho)t} \mathcal{X}(0) + \mathcal{C} \int_{t_f}^{t} e^{\mathcal{A}(\rho)(t-\tau)} \mathcal{F}_i f_i(\tau) d\tau \tag{5.35}$$

where t_f indicates the start time of the i-th fault occurrence. The aim is to find a matrix L such that $\tilde{y}$ belongs to a known and fixed subspace. To this end consider the series expansion of the integrand term in (5.35).

$$\begin{aligned} e^{\mathcal{A}(\rho)(t-\tau)}\mathcal{F}_i &= \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} \left[(A_0 - LC)^k F_i + (A_0 - LC)^{k-1}(A(\rho) - A_0)F_i\right] + \\ &+ \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} \left[\sum_{j=1}^{k-1}(A_0 - LC)^{j-1}(A(\rho) - A_0)A(\rho)^{k-j}F_i\right] \end{aligned} \tag{5.36}$$

It can be seen that there is a term that does not depend on the uncertainty ρ. A possible way to enhance the fault sensitivity is to impose a constraint such that $\tilde{y}$ belongs to a predefined subspace. Let

$$(A_0 - LC)F_i = \lambda F_i \tag{5.37}$$

with $\lambda \in (-\infty, 0)$. Then, (5.35) becomes

$$\begin{aligned} \tilde{y}(t) &\approx C\int_{t_f}^{t} e^{\lambda(t-\tau)}F_i f_i(\tau)d\tau + \\ &+ C\int_{t_f}^{t} e^{(A_0-LC)(t-\tau)}(A_0 - LC)^{-1}(A(\rho) - A_0)F_i f_i(\tau)d\tau + \\ &+ C\int_{t_f}^{t}\sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} \left[\sum_{j=1}^{k-1}(A_0 - LC)^{j-1}(A(\rho) - A_0)A(\rho)^{k-j}F_i\right] f_i(\tau)d\tau \end{aligned} \tag{5.38}$$

Notice that the first term on the right hand side of (5.38) is invariant with respect to ρ, hence, it has a fixed direction in the output subspace. Then the projection matrix H can be chosen in order to project $y(t)$ along the direction CF_i. This can be done by setting

$$H \doteq I - C\hat{F}_i(C\hat{F}_i)^+ \tag{5.39}$$

where

$$(C\hat{F}_i)^+ \doteq \left((C\hat{F}_i)'C\hat{F}_i\right)^{-1}(C\hat{F}_i)' \tag{5.40}$$

is the left pseudo-inverse of $C\hat{F}_i$, which is well defined if the Assumption A-5.4 holds true. The zero frequency gain from fault (f_i) to residual (r) is defined as

$$G \doteq -\frac{1}{\lambda}HCF_i + HC(A_0 - LC)^{-1}(A(\rho) - A_0)A(\rho)^{-1}F_i \tag{5.41}$$

Notice that the first term on the right hand side of (5.41) does not depend on the uncertainty ρ, then the performance measure index is defined by

$$\mu \doteq \frac{\gamma^2}{\|\frac{1}{|\lambda|}HCF_i\|^2} = \frac{\gamma^2\lambda^2}{\|HCF_i\|^2} \tag{5.42}$$

To achieve the best performance in FDI it is advisable to find L such that μ is small. In order to satisfy (5.37) L can be chosen as

$$L = (A_0 - \lambda I)F_i(CF_i)^+ + \bar{L}(I - CF_i(CF_i)^+) \tag{5.43}$$

The optimization problem (5.25) can be rewritten in order to solve the sensitivity optimization problem

$$\begin{array}{ll} \min\limits_{L} & \dfrac{\gamma^2\lambda^2}{\|HCF_i\|^2} \\ \textit{subject to} & \begin{cases} \|r\|_2^2 \leq \gamma^2\|\hat{f}_i\|_2^2 \\ A_0 - LC \quad \textit{quadratically stable} \\ L = (A_0 - \lambda I)F_i(CF_i)^+ + \bar{L}(I - CF_i(CF_i)^+) \end{cases} \end{array} \tag{5.44}$$

Taking into account the eigenvector constraint (5.37), (5.30) can be rewritten as

$$\begin{bmatrix} A(\rho)'P_1 + P_1A(\rho) & (A(\rho) - A_0)'P_2 & P_1\hat{F}_i & 0 \\ * & \mathcal{K}' + \mathcal{K} & P_2\hat{F}_i & (HC)' \\ * & * & -\gamma I & 0 \\ * & * & 0 & -\gamma I \end{bmatrix} \preceq 0 \tag{5.45}$$

with

$$\mathcal{K} \doteq P_2\bar{A} - \bar{K}\bar{C} \tag{5.46}$$

$$\bar{K} \doteq P_2\bar{L} \tag{5.47}$$

$$\bar{F} \doteq A_0 - (A_0 - \lambda I)F_i(CF_i)^+C \tag{5.48}$$

$$\bar{C} \doteq C - CF_i(CF_i)^+C \tag{5.49}$$

Notice that the problem (5.44) is a Bilinear Matrix Inequality (BMI) optimization problem, which is a non-convex problem. A way to solve this kind of problem via LMI optimization techniques is of parameterizing over the variable

λ the constraint (5.45). Then for a fixed λ the problem (5.32) becomes

$$\begin{aligned} \min_{\gamma, P_1, P_2, \bar{K}} \quad & t \\ \textit{subject to} \quad & \begin{cases} (5.45) \\ \begin{bmatrix} t & \lambda\gamma \\ \lambda\gamma & 1 \end{bmatrix} \succ 0 \\ P_1 \succ 0, \quad P_2 \succ 0 \\ \gamma > 0 \end{cases} \end{aligned} \tag{5.50}$$

The solution of (5.50) is an upper-bound of the solution of (5.44). For solvability reasons the next assumption is required

A-5.2.5. the pair $(\bar{A}, \bar{C})$ is detectable.

Otherwise a stable filter could not exist. Notice that the problem (5.50) is feasible if and only if the Assumption A-5.2.5 holds true. Then the matrix L is given by

$$L = (A_0 - \lambda I)F_i(CF_i)^+ + \bar{L}(I - CF_i(CF_i)^+) \tag{5.51}$$

with

$$\bar{L} = P_2^{-1}\bar{K} \tag{5.52}$$

Consider the case where $(A(\rho), B)$ is a stabilizable pair with constant state feedback, then there exists a matrix D such that $A(\rho) + BD$ is quadratically stable. Then, the constraint (5.45) can be rewritten as

$$\begin{bmatrix} (A(\rho)+BD)'P_1 + P_1(A(\rho)+BD) & Q_1\left(A(\rho)-A_0\right)' P_2 Q_1 & \hat{F}_i Q_1 & 0 \\ * & \mathcal{K}' + \mathcal{K} & P_2\hat{F}_i & (HC)' \\ * & * & -\gamma I & 0 \\ * & * & 0 & -\gamma I \end{bmatrix} \preceq 0 \tag{5.53}$$

In this case a solution to the FDI problem can be obtained by solving the problem (5.50) with constraint (5.45) replaced by (5.53).

Proposition 3 *Consider a dynamical system subject to simultaneous fault signals, like (5.1) and let the Assumptions A-5.1-A-5.2.5 be satisfied. Then the problem (5.50) has always a feasible solution.*

5.3 Example

This section explains a practical design problem for a system subject to model uncertainties. Consider the dynamical system described by

$$\begin{cases} \dot{x}(t) &= A(\rho)x(t) + Bu(t) + F_1 f1(t) + F_2 f_2(t) \\ y(t) &= Cx(t) \end{cases} \tag{5.54}$$

where

$$A(\rho) = A_0 + \rho A_1 \tag{5.55}$$

with

$$A_0 = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.5 & 0.6 \\ 0 & -0.6 & -0.5 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad F_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{5.56}$$

where ρ is bounded, with known extremals

$$\rho \in [-1 \quad 1] \tag{5.57}$$

The uncertainty ρ causes a change in the imaginary part of the complex poles of the nominal system . The system is, also, subject to the actuator faults $f_1(t)$ and $f_2(t)$. In order to isolate these faults, a bank of two residual generator is designed. Notice that (5.54) is quadratically stable for all allowable values of ρ. Moreover, the Assumptions A-5.2-A-5.5 hold true. In order to evaluate the proposed design method, the frequency behavior, of the residual generators designed, is analyzed. Consider the following transfer functions for a fixed ρ for the residual generator related to the isolation of the fault $f_1(t)$

$$G_{f_1 r_1}(j\omega) = H_1 \mathcal{C}(j\omega - \mathcal{A}_1(\rho))^{-1}\mathcal{F}_1 \tag{5.58}$$

$$G_{f_2 r_1}(j\omega) = H_1 \mathcal{C}(j\omega - \mathcal{A}_1(\rho))^{-1}\mathcal{F}_2 \tag{5.59}$$

with $\mathcal{A}_1(\rho)$ defined by

$$\mathcal{A}_1(\rho) = \begin{bmatrix} A(\rho) & 0 \\ A(\rho) - A_0 & A_0 - L_1 C \end{bmatrix} \tag{5.60}$$

and the transfer functions for the residual generator related to the isolation of the fault $f_2(t)$

$$G_{f_2r_2}(j\omega) = H_2\mathcal{C}(j\omega - \mathcal{A}_2(\rho))^{-1}\mathcal{F}_2 \tag{5.61}$$

$$G_{f_1r_2}(j\omega) = H_2\mathcal{C}(j\omega - \mathcal{A}_2(\rho))^{-1}\mathcal{F}_1 \tag{5.62}$$

with $\mathcal{A}_2(\rho)$ defined by

$$\mathcal{A}_2(\rho) = \begin{bmatrix} A(\rho) & 0 \\ A(\rho) - A_0 & A_0 - L_2C \end{bmatrix} \tag{5.63}$$

The matrices L_1 and L_2 are obtained as solution of the problem (5.50). Moreover recall the singular value definition

$$S(j\omega) \doteq G^*(j\omega)G(j\omega) \tag{5.64}$$

Fig. 5.1-5.3 show the singular value plot obtained from the two residual

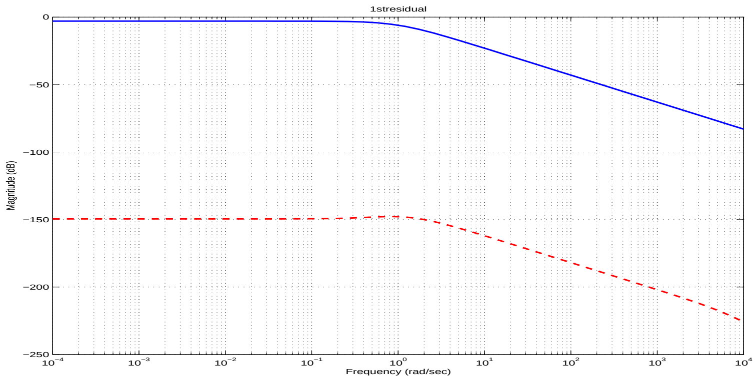
Fig. 5.1(a), 5.2(a) and 5.3(a)	Fig. 5.1(b), 5.2(b) and 5.3(b)
(blue solid line) $S_{f_1r_1}(j\omega)$	(blue solid line) $S_{f_2r_2}(j\omega)$
(red dashed line) $S_{f_2r_1}(j\omega)$	(red dashed line) $S_{f_1r_2}(j\omega)$

Table 5.1. Legend for Fig. 5.1, 5.2 and 5.3

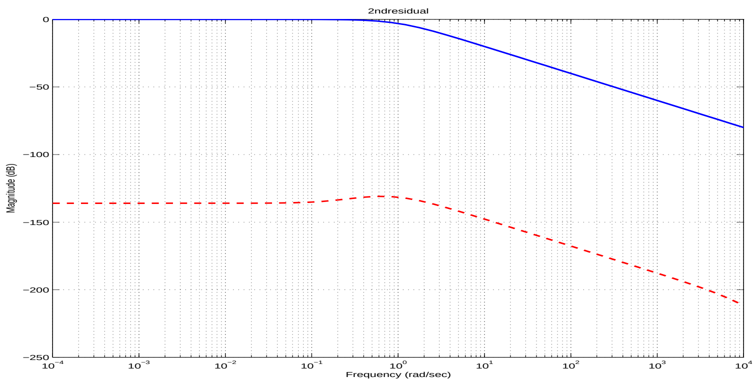
generator. Fig. 5.1(a), 5.2(a) and 5.3(a) illustrate the frequency behavior of the residual generator related to the fault f_1, where the uncertainty is $\rho = 0$, $\rho = 1$ and $\rho = -1$, respectively. Fig. 5.1(b), 5.2(b) and 5.3(b) illustrate the frequency behavior of the residual generator related to the fault f_2. It can be easily verified that each residual generator can reject the nuisances in an efficient way. However, the effects of the uncertainty ρ can be evaluated as follows. It can be seen in [EBSK97] that the disturbances induced by the uncertainty ρ in (5.12) can be evaluated as two additional faults f_{a_1} and f_{a_2} with maps defined by

$$F_{a_1} \doteq A_1F_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.65}$$

$$F_{a_2} \doteq A_1F_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{5.66}$$



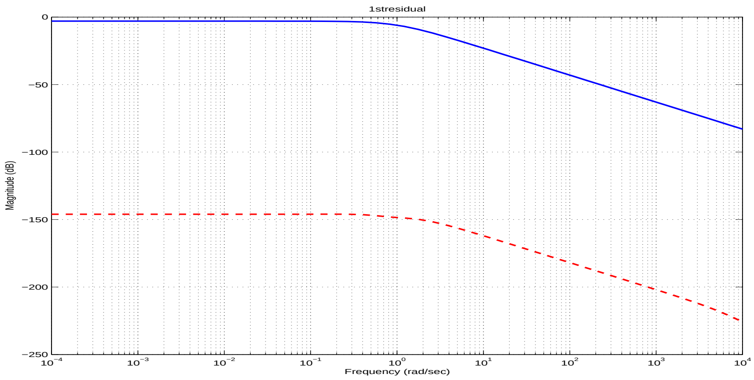
(a) The 1^{st} residual generator



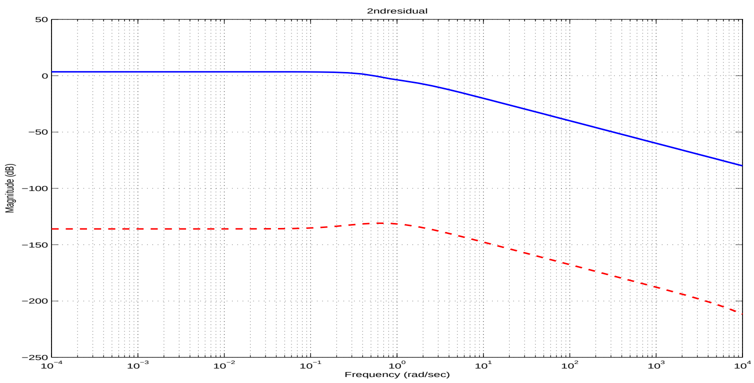
(b) The 2^{nd} residual generator

Figure 5.1. Singular value plot from the nominal system

In order to take into account the effects of the uncertainty, the singular value plot of the transfer functions $G_{f_1 r_1}(j\omega)$ and $G_{f_2 r_2}(j\omega)$ are compared with the singular value plot of the transfer function from f_{a2} to the residuals r_1 and, respectively, r_2. Then, Fig. 5.4(a) illustrates the singular value plot referred to the first residual generator. Notice that in this case the residual generator has a good disturbance rejection. Fig. 5.4(b) illustrates the singular value plot referred to the second residual generator. Notice that for disturbances (ρ) which have a frequency range from zero to 1Hz the residual generator does not have a good rejection of the disturbances induced by the model uncertainties, while for model uncertainties which are time-variant and with frequency greater than 1Hz the residual generator can efficiently reject these induced disturbances.



(a) The 1^{st} residual generator



(b) The 2^{nd} residual generator

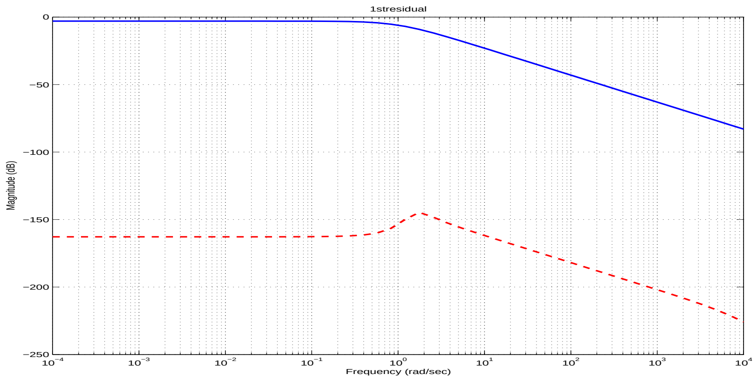
Figure 5.2. Singular value plot for $\rho = 1$

Remark 2 *Notice that the residual generator has a quadratically stable dynamics even in presence of a time-varying uncertainty vector ρ, i.e. if $\rho = \rho(t)$ is a bounded continuous function of time.*

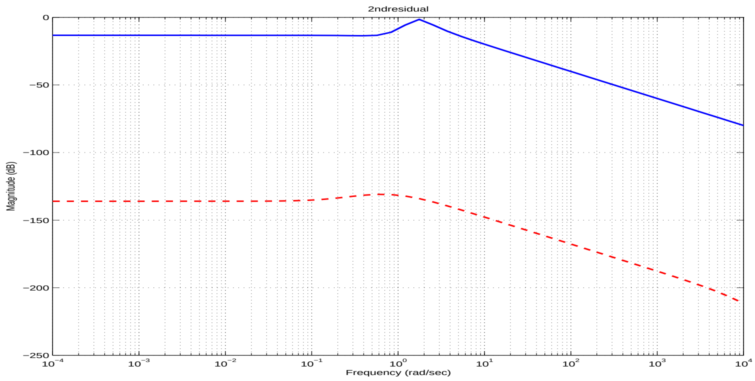
In order to evaluate the effect of model uncertainties in the residual generators filters, a simulation is carried out. It is considered a time-varying disturbance $\rho(t)$ defined by

$$\rho(t) = sin(6.2832t) \tag{5.67}$$

which is a sinusoidal signal with frequency equal to $1\,Hz$. Moreover, the system is subject at $t = 30s$ to a sudden change in the first actuator and at $t = 60s$ in the second actuator, as depicted in Fig. 5.6(a). Fig. 5.6(b) illustrates the



(a) The 1^{st} residual generator



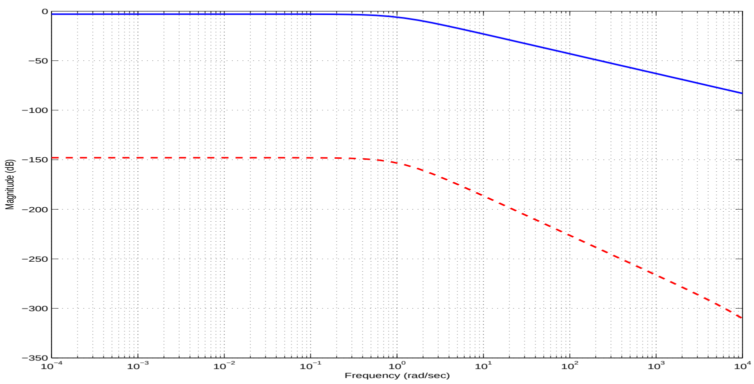
(b) The 2^{nd} residual generator

Figure 5.3. Singular value plot for $\rho = -1$

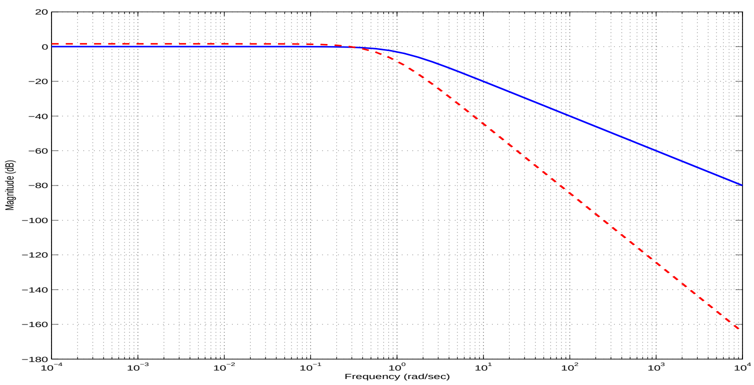
behavior of system outputs. Fig. 5.5 illustrates the output of the residual generators. Notice that the two faults are overlapped for $70s$ and that the two residual generators reject efficiently the nuisance. Notice that the effects of model uncertainties are efficiently rejected by the residual generators. In particular, the first residulal generator can efficiently reject the induced disturbances. The effects of the induced disturbances are more visible in the second residual generator output, recalling that the uncertainty ρ has a frequency at $1Hz$ that is not efficiently rejected, as it can be seen in Fig. 5.4(b).

Fig. 5.4(a)	Fig. 5.4(b)
$S_{f_1r_1}(j\omega)$	$S_{f_2r_2}(j\omega)$
$S_{f_a2r_1}(j\omega)$	$S_{f_a2r_2}(j\omega)$

Table 5.2. Legend for Fig. 5.4



(a) The first residual generator



(b) The second residual generator

Figure 5.4. Singular value plot

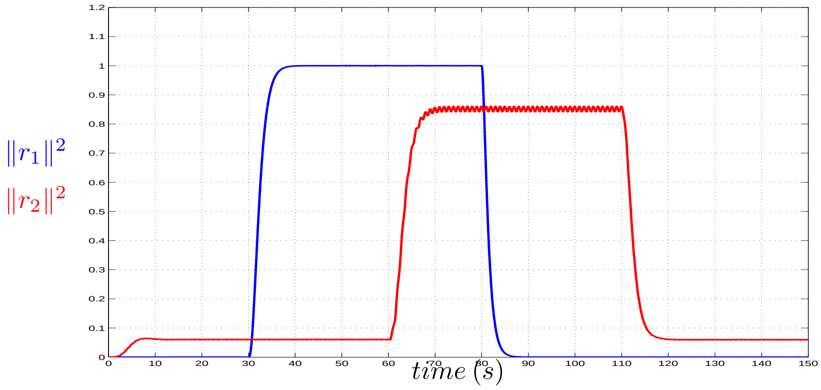
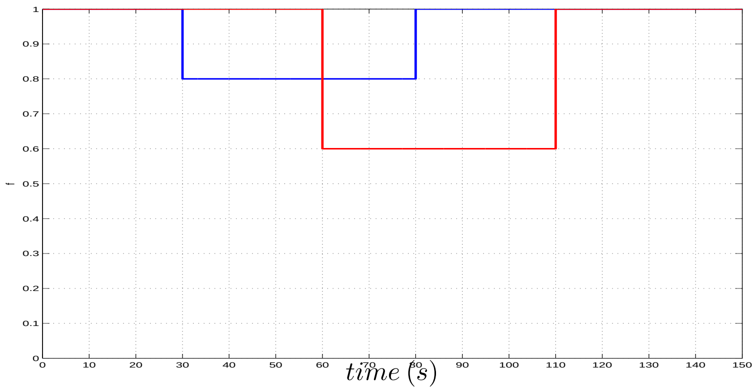
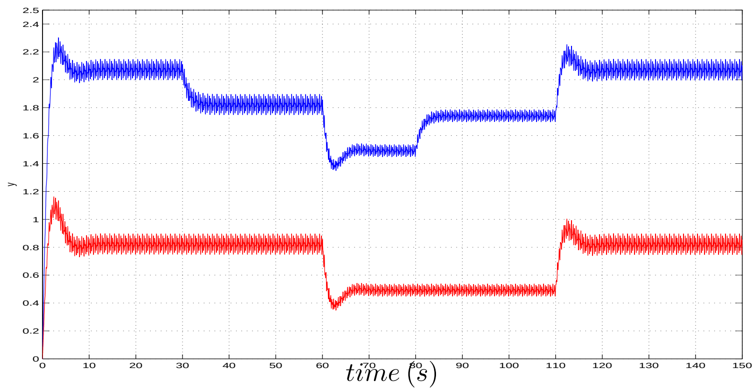


Figure 5.5. The residual generator outputs under faulty condition



(a) The actuator signals with faulty condition



(b) System outputs

Figure 5.6. Simulation signals

5.4 Conclusions

This chapter has presented a solution to the FDI problem for systems subject to model uncertainties, which can be even constant or time-varying. The fault detection capabilities of the residual generator filter are enhanced using an eigen-structure assignment. The filter design leads to an optmization problem. Hence, a performance criterion is introduced in order to select the best filter. The nuisance and disturbance attenuation is performed through an $\mathcal{H}_\infty$ filtering design technique.

Chapter 6

The LPV case

This chapter considers the FDI problem for affine linear parametric variable (ALPV) systems. Linear parameter varying (LPV) modeling techniques have gained a lot of interest, especially those related to vehicle and aerospace control. This approach is particularly appealing because nonlinear plants can be treated as linear systems with a priori not necessarily known but on-line measurable time varying parameters. Moreover, LPV based methods can be considered as an extension of gain scheduling control for nonlinear systems. The residual generators are synthesized by robust filtering design methods under an eigen-structure assignment constraint, in order to enhance the fault detection capability of the filter. The design method leads to the unknown-input observer. While, the disturbances attenuation is solved through an optimal $\mathcal{H}_\infty$ filtering problem. With this geometric constraint it is possible to map the fault injection in a predetermined fixed residual subspace. Moreover, a ratio between the zero frequency gain of the fault to residual map and the $\mathcal{H}_\infty$ gain of the disturbances to residual map is defined as a performance criterion. Then, the minimization of this ratio leads to a residual generator with the maximum fault and the minimum disturbance transmission. In conclusion, the residual generator is obtained as the solution of a family of LMI optimization problems.

Other approaches to this problem can be found in [BBS02]. There, the authors design a residual generator filter using an algebraic approach in combination with an LMI method in order to guarantee the filter stability. The chapter is organized as follows. Section 6.1 introduces the affine linear parameter varying (ALPV) dynamical system. Section 6.2 introduces the fault detection and isolation residual generator (FDIRG) problem with the sensitivity constraint. Section 6.3 treats the necessity of the projection matrix used in the FDIRG problem. Section 6.4 introduces the notion of affine quadratic stability, which is used to find a less-conservative solution to the FDIRG problem. Section 6.5 explains a numerical example for showing the effectiveness of the proposed method. Some conclusions end the paper in Section 6.6.

6.1 The dynamical model

Consider the following ALPV dynamical system subject to r simultaneous different fault signals $f_i(t)$

$$\begin{cases} \dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) + B_w(\rho)w(t) + \sum_{i=1}^{r} F_i f_i(t) \\ y(t) &= Cx(t) + D_n n(t) \end{cases} \tag{6.1}$$

where $\rho = \rho(t)$ is assumed to be a measurable bounded continuously differentiable vector function of the time

$$\rho : [0,\, \infty) \rightarrow \mathbb{R}^d \quad for\; all\; t \geq 0 \tag{6.2}$$

It is assumed that each entries of the vector $\rho = (\rho_1,\, \ldots,\, \rho_d)$ is bounded and ranges between known extremal values $\underline{\rho}_j$ and $\overline{\rho}_j$

$$\rho_i \in \left[\underline{\rho}_j,\, \overline{\rho}_j\right] \tag{6.3}$$

The parameter vector ρ takes value in an hyper-rectangle, with the set Ω of its 2^d vertices defined by

$$\Omega \doteq \left\{(v_1, \ldots, v_d) : \quad v_j \in \{\underline{\rho}_j,\, \overline{\rho}_j\}\right\} \tag{6.4}$$

In (6.1), the matrices $A(\rho), B(\rho), B_w(\rho)$ are defined as follows

$$\begin{aligned} A(\rho) &\doteq A_0 + \sum_{j=1}^{d} \rho_j A_j \\ B(\rho) &\doteq B_0 + \sum_{j=1}^{d} \rho_j B_j \\ B_w(\rho) &\doteq B_0^w + \sum_{j=1}^{d} \rho_j B_j^w \end{aligned} \tag{6.5}$$

where $A_0, \ldots, A_d \in \mathbb{R}^{n\times n}$, $B_0, \ldots, B_d \in \mathbb{R}^{n\times m}$, $B_0^w, \ldots, B_d^w \in \mathbb{R}^{n\times\bar{w}}$ and $\rho_j : [0, \infty) \to \mathbb{R}$ for $j = 1, \ldots, d$ are bounded continuously differentiable functions of the time. Furthermore, it is assumed that the parameter $\rho(t)$ is measurable on-line. The matrices $C \in \mathbb{R}^{p\times n}$, $D_n \in \mathbb{R}^{p\times\bar{n}}$ while $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ indicate the state and, respectively, the input vector; $w(t) \in \mathbb{R}^{\bar{w}}$ and $n(t) \in \mathbb{R}^{\bar{n}}$ the input disturbance and, respectively, the measurement noise vectors signals, while $f_i(t) \in \mathbb{R}$ and $F_i \in \mathbb{R}^n$ for $i = 1, \ldots, r$ represent the r faults signals and, respectively, the fault injection matrices. It is assumed that the fault injection matrices F_i have fixed direction in the state subspace, then w.l.o.g. they can be assumed independent from the parameter ρ. The fault detection of multiple simultaneous faults, is carried out by the design of a bank of residual generators, each one capable of isolating a single fault occurrence. In the following, will be explained the identification and isolation of a single fault occurrence. In order to detect and isolate the i-th fault occurrence, it is advisable to design an FDI filter that satisfies the following properties:

P-6.1. Whenever $f_i(t) \equiv 0$, the map from nuisances (u, n, w and f_j for $j \neq i$) to the residual (r) has an $\mathcal{H}_\infty$-gain predetermined, for all $\rho(t)$.

P-6.2. Whenever $f_i(t) \neq 0$, the zero frequency gain from the fault (f_i) to the residual (r) has a predetermined minimum euclidean norm, for all $\rho(t)$ and irrespective of $f_j(t) \neq 0$ with $j \neq i$.

For the sake of notational simplicity (6.1) is rewritten as

$$\begin{cases} \dot{x}(t) = A(\rho)x(t) + B(\rho)u(t) + B_w(\rho)w(t) + F_i f_i(t) + \hat{F}_i \hat{f}_i(t) \\ y(t) = Cx(t) + D_n n(t) \end{cases} \tag{6.6}$$

where

$$\hat{F}_i \doteq [F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_r] \in \mathbb{R}^{n\times r} \tag{6.7}$$

and

$$\hat{f}_i(t) \doteq [f_1(t),\, \ldots,\, f_{i-1}, f_{i+1}(t),\, \ldots,\, f_r(t)]' \in \mathbb{R}^r \tag{6.8}$$

with $\hat{f}_i(t)$ indicating a nuisance signal. In order to find a solution to the FDI problem for an ALPV system, it is needed to impose some assumptions over the system (6.1). The following assumptions are required:

A-6.1. The pair $(A(\rho),\, C)$ is uniformly observable for all possible trajectories of $\rho(t)$.

A-6.2. All F_i are monic, for $i = 1, \ldots, r$, viz. if $f_i(t) \neq 0$ then $F_i f_i(t) \neq 0$.

A-6.3. The conditions $CF_i \neq 0$, for $i = 1, \ldots, r$, hold true.

A-6.4. The $\text{rank}(C[F_1,\, \ldots,\, F_r]) = r$.

Assumption A-6.1 is required for solvability reasons. That is, if the system (6.1) is uniformly observable, the state of the system may be determined from observations of the inputs and outputs at any time. Assumptions A-6.2 and A-6.3 are related to the *input observability* property. Assumption A-6.4 is related to the *output separability* property.

6.2 The FDI Problem with a Sensitivity Constraint

In order to design the residual generator for the i-th fault occurrence of the ALPV system (6.1) , the following observer is defined

$$\begin{cases} \dot{\hat{x}}(t) &= A(\rho)\hat{x}(t) + B(\rho)u(t) + L(\rho)(y(t) - C\hat{x}(t)) \\ \hat{y}(t) &= C\hat{x}(t) \end{cases} \tag{6.9}$$

where $L(\rho)$ is the observer gain matrix to be determined, which is affine in the parameter ρ.

$$L(\rho) \doteq L_0 + \sum_{j=1}^{d} L_j \rho_j \tag{6.10}$$

with $L_j \in \mathbb{R}^{n\times p}$, $j = 0, \ldots, d$ while at time instant t, $\hat{x}(t) \in \mathbb{R}^n$ is the observer state and $\hat{y}(t) \in \mathbb{R}^p$ is the output estimation. Define the state error as

$$e(t) \doteq x(t) - \hat{x}(t) \tag{6.11}$$

Then, the error dynamics can be written as

$$\dot{e}(t) = A_{cl}(\rho)e(t) + B_w(\rho)w(t) - L(\rho)D_n n(t) + F_i f_i(t) + \hat{F}_i \hat{f}_i(t) \tag{6.12}$$

and the output error $\tilde{y}$ is defined by

$$\tilde{y}(t) \doteq y(t) - \hat{y}(t) = Ce(t) + D_n n(t) \tag{6.13}$$

The residual is defined as the projection of $\tilde{y}$

$$r(t) \doteq H\tilde{y}(t) \tag{6.14}$$

via a projection matrix $H \in \mathbb{R}^{p\times p}$ to be defined. The use of a projection matrix is justified by the fact that the residual is designed in such a way that $\tilde{y}$ belongs to a fixed and known subspace in the presence of the i-th fault occurrence. The dynamical model of the residual generator can be described by the following equations

$$\begin{cases} \dot{e}(t) &= A_{cl}(\rho)e(t) + B_w(\rho)w(t) - L(\rho)D_n n(t) + F_i f_i(t) + \hat{F}_i \hat{f}_i(t) \\ \tilde{y}(t) &= Ce(t) + D_n n(t) \\ r(t) &= H\tilde{y}(t) \end{cases} \tag{6.15}$$

In order to detect the i-th fault occurrence it is necessary that the fault f_i is input observable from the output $\tilde{y}$ and then from the residual r. The following lemma introduces the terms needed by the system (6.15) such that f_i is input observable.

Lemma 13 *Let the Assumptions A-6.2, A-6.3 hold true then there exists a matrix $L(\rho)$ such that $f_i(t)$ is input observable from $\tilde{y}(t)$ and $A_{cl}(\rho)$ is quadratically stable.* □

Proof. Consider

$$\begin{cases} \dot{e}(t) &= A_{cl}(\rho)e(t) + F_i f_i(t) \\ \tilde{y}(t) &= Ce(t) \end{cases} \tag{6.16}$$

By contradiction. Suppose that $f_i(t) \neq 0$ then for the input observability property $\tilde{y}(t) \neq 0$ whatever is ρ. Let assumption A-6.2 holds true, then $f_i(t) \neq 0$ implies that $F_i f_i(t) \neq 0$. Let assumption A-6.3 holds true, then the smallest $(C, A_{cl}(\rho))$-invariant subspace is equal to

$$\underline{\mathcal{W}}(F_i) = \mathcal{I}(F_i) \tag{6.17}$$

and

$$\underline{\mathcal{W}}(F_i) \nsubseteq \mathcal{N}(C) \tag{6.18}$$

Let $L(\rho)$ be such that $A_{cl}(\rho)$ is quadratically stable, then all the trajectories of $e(t)$ converge to $\underline{\mathcal{W}}(F_i)$ whatever is $e(0)$ for all trajectories of ρ. The subspace $\underline{\mathcal{W}}(F_i)$ is in the output observability subspace. Moreover, if $e(0) = 0$ all the trajectories belong to $\underline{\mathcal{W}}(F_i)$. In conclusion the input f_i is input observable. ■
The residual generator matrices $L(\rho)$ and H should be designed in a such way that

$$\begin{cases} r \approx 0 & f_i(t) \equiv 0 \\ r \neq 0 & f_i(t) \neq 0 \end{cases} \tag{6.19}$$

whatever $\hat{f}_i(t)$, $w(t)$ and $n(t)$ and for all trajectories ρ. To this end, a preliminary nuisance attenuation problem is introduced

$$\begin{aligned} &\min_{L(\rho)} \gamma^2 \\ &\quad s.t. \quad \begin{cases} \|r\|_2^2 \leq \gamma^2(\|\hat{f}_i\|_2^2 + \|n\|_2^2 + \|w\|_2^2) \\ A_{cl}(\rho) \; quadratically \; stable \end{cases} \end{aligned} \tag{6.20}$$

The problem is that of finding a matrix $L(\rho)$ that stabilizes (6.12) and minimizes the $\mathcal{H}_\infty$-gain γ from the nuisances and disturbances ($\hat{f}_i$, w and n) to the residual (r), for all the possible trajectories $\rho(t)$. In order to solve the optimization problem (6.20) via LMI techniques, consider the following Lyapunov function

$$V(x) = x'Px \tag{6.21}$$

where P is a positive definite matrix. Then, (6.15) is quadratically stable with an $\mathcal{H}_\infty$-gain γ if the following inequality is satisfied

$$\frac{dV(x)}{dt} + r_i'r_i - \gamma^2(\hat{f}_i'\hat{f}_i + n'n + w'w) < 0 \tag{6.22}$$

for all trajectories ρ and for all signals $\hat{f}_i$, w, n laying in the $\mathcal{L}_2$ functional space. A sufficient condition for (6.22) is the feasibility of the following LMI constraints

$$\begin{cases} \begin{bmatrix} A_{cl}(\rho)'P + PA_{cl}(\rho) & P\hat{F}_i & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_{cl}(\rho)'P + PA_{cl}(\rho) & PB_w(\rho) & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_{cl}(\rho)'P + PA_{cl}(\rho) & -PL(\rho)D_n & (HC)' \\ * & -\gamma I & (HD_n)' \\ * & * & -\gamma I \end{bmatrix} \prec 0 \end{cases} \tag{6.23}$$

to be jointly satisfied by a single positive definite matrix $P = P' \succeq 0$ and for all allowable trajectories of $\rho(t)$. Notice that (6.23) is not tractable, because it involves an infinite number of constraints. Thanks to the affine linear parameter dependence of $A_{cl}(\rho)$, which implies that the set of constraints is convex over all the trajectories of ρ, it is possible to reduce to a finite set the above constraints. Thus, it is enough to verify (6.23) over the vertices $v \in \Omega$.

Remark 3 *Recall that (6.23), in the case of ρ constant and known, reduces to the Bounded Real Lemma, [BGFB94].*

The nuisance attenuation problem (6.20) can be rewritten as follows

$$\begin{aligned} \min_{\gamma, L(\rho)} \quad & \gamma^2 \\ s.t. \quad & \begin{cases} \begin{bmatrix} A_{cl}(v)'P+PA_{cl}(v) & P\hat{F}_i & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_{cl}(v)'P+PA_{cl}(v) & PB_w(v) & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_{cl}(v)'P+PA_{cl}(v) & -PL(v)D_n & (HC)' \\ * & -\gamma I & (HD_n)' \\ * & * & -\gamma I \end{bmatrix} \prec 0 \\ P = P' \succ 0 \\ \text{for all } v \in \Omega \end{cases} \end{aligned} \tag{6.24}$$

Remark 4 *If $\dot{\rho}$ is bounded and known the use of a constant matrix P in the Lyapunov function might be too conservative. A way to reduce such a conservativeness is to resort to parameterized Lyapunov functions, [GAC96].*

The solution of the optimization problem (6.24) is an affine parametric matrix $L(\rho)$ which satisfy the property P-6.1. However, in order to satisfy the property P-6.2, it is necessary to set another constraint over the matrix $L(\rho)$. To this end, consider the response of (6.12) from $f_i(t)$ when $\hat{f}_i \equiv 0$, $n \equiv 0$ and $w \equiv 0$. From Assumption A-6.1 there exists $L(\rho)$ such that the following ALPV system is quadratically stable.

$$\begin{cases} \dot{e} &= A_{cl}(\rho)e + F_i f_i \\ \tilde{y} &= Ce \end{cases} \tag{6.25}$$

Rewrite $\tilde{y}(t)$ as

$$\tilde{y}(t) = C\Phi_\rho(t, t_f)e(0) + C\int_{t_f}^{t} \Phi_\rho(t,\tau)F_i f_i(\tau)d\tau \tag{6.26}$$

where t_f is the start time of the i-th fault occurrence. The state transition matrix $\Phi_\rho(t, t_f)$ is defined as the solution of

$$\begin{cases} \dot{\Phi}_\rho(t, t_f) &= A_{cl}(\rho)\Phi_\rho(t, t_f) \\ \Phi_\rho(t_f, t_f) &= I \end{cases} \tag{6.27}$$

In order to enhance the fault sensitivity, it is advisable to force $\tilde{y}(t)$ to converge to a fixed subspace. To this end consider the following eigenvalue-eigenvector constraint

$$A_{cl}(\rho)F_i = \lambda F_i \quad \lambda \neq 0 \in \mathbb{R} \tag{6.28}$$

on the closed-loop dynamic for all trajectories of ρ. A suitable choice for $L(\rho)$ that satisfies (6.28) is

$$L(\rho) = (A(\rho) - \lambda I)F_i(CF_i)^+ + \bar{L}(\rho)(I - (CF_i)(CF_i)^+) \tag{6.29}$$

with $\bar{L}(\rho) \in \mathbb{R}^{n\times p}$ of the form

$$\bar{L}(\rho) \doteq \bar{L}_0 + \sum_{j=1}^{d} \bar{L}_j \rho_j \tag{6.30}$$

to be determined in such a way that $A_{cl}(\rho)$ is quadratically stable and the $\mathcal{H}_\infty$-gain γ is minimized. Obviously, λ has to be chosen in $(-\infty, 0)$, for stability reasons. In (6.29), $(CF_i)^+$ denotes the left pseudo-inverse of CF_i, viz.

$$(CF_i)^+ = ((CF_i)'CF_i)^{-1}(CF_i)' \tag{6.31}$$

which always exists thanks to Assumption A-6.3. Using the Peano-Baker formula $\Phi_\rho(t, t_f)$ can be expressed with respect to $A_{cl}(\rho)$ as follows

$$\begin{array}{l} \Phi_\rho(t, t_f) = I + \int_0^t A_{cl}(\rho(\tau_1))d\tau_1 + \\ + \int_{t_f}^t \int_{t_f}^{\tau_1} A_{cl}(\rho(\tau_1))A_{cl}(\rho(\tau_2))d\tau_2 d\tau_1 + \cdots + \\ \int_{t_f}^t \cdots \int_{t_f}^{\tau_{n-1}} A_{cl}(\tau_1)\cdots A_{cl}(\tau_n)d\tau_n \cdots d\tau_1 + \cdots \end{array} \tag{6.32}$$

Then, considering the integrand part of (6.26) and taking into account the constraint (6.28) and the formula (6.32)

$$\Phi_\rho(t, \tau)F_i = F_i + \lambda(t-\tau)F_i + \lambda^2 \frac{(t-\tau)^2}{2!}F_i + \cdots + \lambda^n \frac{(t-\tau)^n}{n!}F_i + \cdots \tag{6.33}$$

which can also be rewritten as

$$\Phi_\rho(t, \tau)F_i = e^{\lambda(t-\tau)}F_i \tag{6.34}$$

With this relation we can rewrite (6.26) as

$$\tilde{y}(t) = C\Phi_\rho(t, t_f)e(0) + C\int_{t_f}^t e^{\lambda(t-\tau)}F_i f_i(\tau)d\tau \tag{6.35}$$

Notice that with (6.28) the fault signal f_i is mapped in the output subspace generated by the columns of CF_i irrespective of $\rho(t)$. The zero frequency gain from the fault f_i to the residual r is given by

$$G_\rho^i(0) \doteq \frac{1}{|\lambda|}HCF_i \tag{6.36}$$

Define now the sensitivity of the residual from fault f_i as the minimum singular value of the zero frequency gain

$$\underline{\sigma}(G_\rho^i(0)) = \frac{1}{|\lambda|}\|HCF_i\| \tag{6.37}$$

where $\|HCF_i\|$ indicates the euclidean norm of the vector HCF_i. Higher values of (6.37) are related to a better fault sensitivity properties. Then, in order to obtain the maximum nuisance attenuation level while guaranteeing a minimum level of fault sensitivity the following ratio is introduced

$$\mu \doteq \frac{\gamma^2}{\underline{\sigma}^2(G_\rho^i(0))} \tag{6.38}$$

The solution to the FDI problem is to find a matrix $L(\rho)$ and a matrix H such that (6.12) is quadratically stable and (6.38) is minimized. To this end, let the following problem be introduced

$$\min_{\gamma,\,\lambda,\,\bar{L}_0,\dots,\bar{L}_d} \quad t$$
$$s.t. \quad \begin{cases} \|r_i\|_2^2 \leq \gamma^2(\|\hat{f}_i\|_2^2 + \|n\|_2^2 + \|w\|_2^2) \\ A_{cl}(\rho)\ \textit{quadratically stable} \\ L(\rho) = (A(\rho) - \lambda I)F_i(CF_i)^+ + \bar{L}(\rho)(I - (CF_i)(CF_i)^+) \\ \dfrac{\lambda^2\gamma^2}{\|HCF_i\|^2} \leq t \end{cases} \tag{6.39}$$

Notice that (6.39) is a Bilinear Matrix Inequalities (BMI) optimization problem, which is a non-convex optimization problem. In order to solve (6.39) through LMI techniques it is chosen to parameterize the variable λ. In fact, for λ fixed (6.39) becomes an LMI optimization problem. Then, using (6.29), (6.24) can be rewritten for a fixed value of λ as

$$\min_{t,P,K_0,\dots,K_d} \quad t$$
$$s.t. \quad \begin{cases} \begin{bmatrix} \mathcal{A}(v) & P\hat{F}_i & (HC)' \\ * & -\gamma & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} \mathcal{A}(v) & PB_w(v) & (HC)' \\ * & -\gamma & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} \mathcal{A}(v) & \mathcal{K}(v) & (HC)' \\ * & -\gamma & (HD_n)' \\ * & * & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} t & \lambda\gamma \\ \lambda\gamma & 1 \end{bmatrix} \succeq 0 \\ P = P' \succ 0 \\ \gamma > 0 \\ \textit{for all}\, v \in \Omega \end{cases} \tag{6.40}$$

where

$$\begin{aligned}\mathcal{A}(v) &\doteq \bar{A}'(v)P + P\bar{A}(v) + \\ &\quad -\bar{C}_i'\left(K_0 + \sum_{j=1}^{d} v_j K_j\right)' - \left(K_0 + \sum_{j=1}^{d} v_j K_j\right)\bar{C}_i \qquad (6.41)\\ \mathcal{K}(v) &\doteq -\left(K_0 + \sum_{j=1}^{d} v_j K_j\right) D_n \qquad (6.42)\end{aligned}$$

and

$$\begin{aligned}\bar{A}(v) &\doteq A(v) - (A(v) - \lambda I)F_i(CF_i)^+C \qquad (6.43)\\ \bar{C}_i &\doteq I - (CF_i)(CF_i)^+ \qquad (6.44)\\ K_j &\doteq P\bar{L}_j \qquad (6.45)\end{aligned}$$

Moreover, for the solvability of (6.40), along with Assumptions A-6.1-A-6.4, it is also assumed that

A-6.5. the pair $(\bar{A}(\rho), \bar{C}_i)$ is uniformly observable for all ρ.

Notice that in (6.40), the slack variable t is introduced to implement the constraint

$$\lambda^2\gamma^2 \leq t \qquad (6.46)$$

as an LMI. Finally, it can be shown that the matrices $\bar{L}_j$ are obtained in the form

$$\bar{L}_j = P^{-1}K_j \qquad (6.47)$$

In conclusion the affine matrix $L(\rho)$ is obtained by solving the problem (6.40) over a grid of values for λ and choosing, as a solution, the one at which the ratio (6.38) is minimal.

Remark 5 *Notice that λ is also a mode for the error dynamics (6.12). The error dynamic will be slower as λ is chosen closer to zero.*

Remark 6 *Notice that the use of the same matrix P in (6.40) for all the three matrix constraints could be too restrictive. A way to reduce this kind of conservativeness is to find a way to use the results exposed in [Sch00].*

Proposition 4 *Consider the ALPV system (6.1) and let assumptions A-6.1-A-6.5 be satisfied then, (6.40) has solution for any fixed $\lambda \in (-\infty, 0)$.* □

6.3 The Projection Matrix

The previous section explains how with a proper choice of $L(\rho)$ the residual converges to a predefined subspace in the case of the i-th fault occurrence. Then, it is advisable to project $\tilde{y}$ over such subspace. To this end the projection matrix H can be chosen as follows. Let the reachability matrix of the system (6.25) be

$$\begin{aligned} \mathbf{R} &= [p_0, \ldots, p_n - 1] \\ p_{k+1} &= -A_{cl}(\rho)p_k + \dot{p}_k \\ p_0 &= F_i \end{aligned} \tag{6.48}$$

Then, thanks to the eigenvector assignment constraint (6.28), the reachability matrix for the system (6.25) is fixed and becomes

$$\mathbf{R} = [F_i \quad -\lambda F_i \quad \cdots \quad (-1)^{n-1}\lambda^{n-1}F_i] \tag{6.49}$$

which obviously has rank one. Moreover it can be shown that $\Im(\mathbf{R})$ is also a parameter-varying $(C, A(\rho))$-invariant subspace that contains the image of F_i, [BBS03]. Then, the projection matrix can be defined as follows

$$H \doteq I - C\hat{F}_i(C\hat{F}_i)^+ \tag{6.50}$$

The reachability matrix $\mathbf{R}$ will be mapped in the output space through the matrix C as follows

$$C\mathbf{R} = [CF_i \quad -\lambda CF_i \quad \cdots \quad (-1)^{n-1}\lambda^{n-1}CF_i] \tag{6.51}$$

which obviously has rank one. Assumption A-6.4 can now be clarified. In the case where there are r faults to detect it is required that the detection space (6.51) of each fault in the output space will have to be pairwise disjoint. Consider two distinct faults i and j. Then, if CF_i and CF_j are co-linear the two faults cannot be isolated. In conclusion, the faults that do not satisfy A-6.4 can only be detected but not isolated from the others.

6.4 The Affine Quadratic Stability case

This section explains the design of the FDI filter with sensitivity constraint when both $\rho(t)$ and $\dot{\rho}(t)$ are measurable and bounded functions of time. It is assumed that each entry $\dot{\rho}_i$ of the vector $\dot{\rho}$ has known lower and upper bounds

$$\dot{\rho}_i \in \left[\underline{\varrho}_i, \overline{\varrho}_i\right] \tag{6.52}$$

Then $\dot{\rho}$ takes values in an hyper-rectangle, with the set $\mathcal{D}\Omega$ of its vertices defined by

$$\mathcal{D}\Omega = \left\{(k_1, \ldots, k_d) : \quad k_i \in \{\underline{\varrho}_i\, \overline{\varrho}_i\}\right\} \tag{6.53}$$

The major advantage of the knowledge of $\dot{\rho}$ is the possibility to reduce the conservatism of quadratic stability required for the error dynamics (6.15), by requiring that (6.15) is affine quadratically stable. To this end the following affine quadratic Lyapunov function is introduced

$$V(x, \rho) \doteq x' P(\rho) x, \quad P(\rho) \doteq P_0 + \rho_1 P_1 + \cdots \rho_N P_N \tag{6.54}$$

with $V(x, \rho) > 0$ and $\frac{dV(x,\rho)}{dt} < 0$ along all possible trajectories of ρ. The requirement $V(x, \rho) > 0$ is equivalent to

$$P(\rho) > 0 \tag{6.55}$$

while $\frac{dV(x,\rho)}{dt} < 0$ is equivalent to

$$A_{cl}(\rho)' P(\rho) + P(\rho) A_{cl}(\rho) + P(\dot{\rho}) - P_0 < 0 \tag{6.56}$$

because $\frac{dP(\rho)}{dt} = P(\dot{\rho}) - P_0$. In order to have a finite set of LMI constraints satisfying (6.56) it is needed to restrict the choice of the affine Lyapunov matrix $P(\rho)$. Next theorem introduces a test for affine quadratic stability.

Theorem 6 *[GAC96] Consider the affine linear parametric system (6.1). Let Ω and $\mathcal{D}\Omega$ denote the sets of corners or the parameter box and the rate-of-variation box, respectively, and let*

$$\rho_{mean} \doteq \left(\frac{\underline{\rho}_1 + \bar{\rho}_1}{2}, \cdots, \frac{\underline{\rho}_N + \bar{\rho}_N}{2}\right) \tag{6.57}$$

denote the average value of the parameter vector. This system is affine quadratically stable if $A(\rho_{mean})$ is stable and there exist $N+1$ symmetric matrices $P_0, \ldots, P_N$ such that

$$P(\rho) \doteq P_0 + \rho_1 P_1 + \cdots \rho_N P_N \tag{6.58}$$

satisfying

$$\begin{aligned} &A_{cl}(v)'P(v) + P(v)A_{cl}(v) + P(k) - P_0 < 0 \\ &A_i'P_i + P_iA_i \geq 0 \quad \textit{for } i = 1, \ldots, N \end{aligned} \tag{6.59}$$

for all $(v, k) \in \Omega \times \mathcal{D}\Omega$. When (6.59) are feasible, a Lyapunov function for this system and for all trajectories ρ is given by

$$V(\rho, x) \doteq x'P(\rho)x \tag{6.60}$$

System (6.15) is required to be affine quadratically stable and with an $\mathcal{H}_\infty$-gain γ, these properties are fulfilled if the following inequality

$$\frac{dV(x,\rho)}{dt} + r_i'r_i - \gamma^2(\hat{f}_i'\hat{f}_i + n'n + w'w) < 0 \tag{6.61}$$

holds true for all the trajectories of ρ and whatever are $\hat{f}_i$, n, $w \in \mathcal{L}_2$. A sufficient condition for (6.61) is that the following LMI's are feasible

$$\begin{cases} \begin{bmatrix} A_{cl}(\rho)'P(\rho) + P(\rho)A_{cl}(\rho) + P(\dot{\rho}) - P_0 & P(\rho)\hat{F}_i & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_{cl}(\rho)'P(\rho) + P(\rho)A_{cl}(\rho) + P(\dot{\rho}) - P_0 & P(\rho)B_w(\rho) & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_{cl}(\rho)'P(\rho) + P(\rho)A_{cl}(\rho) + P(\dot{\rho}) - P_0 & -P(\rho)L(\rho)D_n & (HC)' \\ * & -\gamma I & (HD_n)' \\ * & * & -\gamma I \end{bmatrix} \prec 0 \\ P(\rho) \doteq P_0 + \rho_1 P_1 + \cdots + \rho_N P_N \succ 0 \end{cases} \tag{6.62}$$

for all possible trajectories of ρ. The constraints (6.62) require an infinite number of tests, which can be reduced to a finite set of LMI's constraint, thanks to the affine structure and by imposing that each constraint in (6.62) is multi-convex, i.e. each constraint is convex along each direction ρ_i of the parameter space. Then

$$\begin{cases} \begin{bmatrix} A_{cl}(v)'P(v)+P(v)A_{cl}(v)+P(k)-P_0 & P(v)\hat{F}_i & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_{cl}(v)'P(v)+P(v)A_{cl}(v)+P(k)-P_0 & P(v)B_w(v) & (HC)' \\ * & -\gamma I & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_j^{cl}P_j+P_jA_j^{cl} & P_jB_j^w \\ * & 0 \end{bmatrix} \succeq 0 \\ \begin{bmatrix} A_{cl}(v)'P(v)+P(v)A_{cl}(v)+P(k)-P_0 & -P(v)L(v)D_n & (HC)' \\ * & -\gamma I & (HD_n)' \\ * & * & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} A_j^{cl}P_j+P_jA_j^{cl} & -P_iL_jD_n \\ * & 0 \end{bmatrix} \succeq 0 \\ i=1,\dots,N \quad v\in\Omega \quad k\in\mathcal{D}\Omega \end{cases} \tag{6.63}$$

where

$$A_{cl}(v) = A_0^{cl} + \sum_{j=1}^{d} v_j A_j^{cl} \tag{6.64}$$

Theorem 7 *[GAC96] Consider the system with affine dependence on ρ and assume that $A(\rho_{mean})$ is quadratically stable. Then if there exist $N+1$ matrices $P_0,\dots,P_N$ such that (6.63) hold true, the system has affine quadratic $\mathcal{H}_\infty$ performance γ.*

In order to take into account the affine quadratic stability in filter design the following optimization problem is introduced

$$\min_{\gamma,\,\lambda,\,\bar{L}_0,\dots,\bar{L}_d} \quad t$$
$$\text{s.t.} \begin{cases} \|r_i\|_2^2 \le \gamma^2(\|\hat{f}_i\|_2^2+\|n\|_2^2+\|w\|_2^2) \\ A_{cl}(\rho) \quad \textit{affine quadratically stable} \\ L(\rho) = (A(\rho)-\lambda I)F_i(CF_i)^+ + \bar{L}(\rho)(I-(CF_i)(CF_i)^+) \\ \dfrac{\lambda^2\gamma^2}{\|HCF_i\|^2} \le t \end{cases} \tag{6.65}$$

Then, taking into account the eigenvector constraint and (6.63) the previous optimization problem becomes

$$\min_{t,P_0,\ldots,P_d,K_0,\ldots,K_d} \quad t$$
$$s.t. \quad \begin{cases} \begin{bmatrix} \mathcal{A}(v) & P(v)\hat{F}_i & (HC)' \\ * & -\gamma & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} \mathcal{A}(v) & P(v)B_w(v) & (HC)' \\ * & -\gamma & 0 \\ * & 0 & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} \mathcal{A}(v) & \mathcal{K}(v) & (HC)' \\ * & -\gamma & (HD_n)' \\ * & * & -\gamma I \end{bmatrix} \prec 0 \\ \begin{bmatrix} \mathcal{A}_j'P_j + P_j\mathcal{A}_j & P_jB_j^w \\ * & 0 \end{bmatrix} \succeq 0 \\ \begin{bmatrix} \mathcal{A}_j'P_j + P_j\mathcal{A}_j & \mathcal{K}_j \\ * & 0 \end{bmatrix} \succeq 0 \\ \begin{bmatrix} t & \lambda\gamma \\ \lambda\gamma & 1 \end{bmatrix} \succeq 0 \\ \gamma > 0 \\ \textit{for all}\, v \in \Omega \quad k \in \mathcal{D}\Omega \\ j = 1, \ldots, d \end{cases} \tag{6.66}$$

where

$$\mathcal{A}(v) \doteq \bar{A}'(v)P(v) + P(v)\bar{A}(v) +$$
$$-\bar{C}_i'\left(K_0 + \sum_{j=1}^{d} v_jK_j\right)' - \left(K_0 + \sum_{j=1}^{d} v_jK_j\right)\bar{C}_i \tag{6.67}$$
$$\mathcal{K}(v) \doteq -K_0D_n - \sum_{j=1}^{d} v_jK_jD_n \tag{6.68}$$
$$\mathcal{A}_j \doteq A_j(I - F_i(CF_i)^+C) - \bar{L}_j\bar{C}_iC \tag{6.69}$$
$$\mathcal{K}_j \doteq -P_jA_jK_i(CF_i)^+C - P_j\bar{L}_j\bar{C}_iC \tag{6.70}$$

and

$$\bar{A}(v) \doteq A(v) - (A(v) - \lambda I)F_i(CF_i)^+C \quad (6.71)$$

$$\bar{C}_i \doteq I - (CF_i)(CF_i)^+ \quad (6.72)$$

$$K_j \doteq P\bar{L}_j \quad (6.73)$$

6.5 Example

This section explains the FDI filter design method for ALPV systems through a practical example. Consider the dynamical system described by

$$\begin{cases} \dot{x}(t) = A(\rho)x(t) + Bu(t) + F_1 f1(t) + F_2 f_2(t) \\ y(t) = Cx(t) \end{cases} \quad (6.74)$$

where

$$A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 \quad (6.75)$$

with

$$A_0 = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.5 & 0.6 \\ 0 & -0.6 & -0.5 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad F_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (6.76)$$

where $\rho = [\rho_1 \quad \rho_2]'$ is bounded, with known extremals

$$\rho_1 \in [-1 \quad 1] \quad (6.77)$$

$$\rho_2 \in [-.4 \quad .4] \quad (6.78)$$

Notice that (6.74) is quadratically stable, moreover the Assumptions A-6.1-A-6.4 are satisfied. The system is subject to actuator faults, indicated with f_1 and, respectively, f_2. In order to isolate these faults a bank of two residual generators is designed. The frequency behavior is analyzed in order to show the validity of the proposed method. Consider a fixed value of the uncertainty vector, $\bar{\rho}$. Define the transfer function from fault f_1 to the residual r_1 as

$$G_{f_1,r_1}(j\omega) = H_1 C(j\omega - A(\bar{\rho}) + L_1(\bar{\rho})C)^{-1} F_1 \tag{6.79}$$

$$G_{f_2,r_1}(j\omega) = H_1 C(j\omega - A(\bar{\rho}) + L_1(\bar{\rho})C)^{-1} F_2 \tag{6.80}$$

and the transfer function from fault f_2 to the residual r_2 as

$$G_{f_2,r_2}(j\omega) = H_2 C(j\omega - A(\bar{\rho}) + L_2(\bar{\rho})C)^{-1} F_2 \tag{6.81}$$

$$G_{f_1,r_2}(j\omega) = H_2 C(j\omega - A(\bar{\rho}) + L_2(\bar{\rho})C)^{-1} F_1 \tag{6.82}$$

Then, the singular value is defined by

$$S(j\omega) \doteq G^*(j\omega)G(j\omega) \tag{6.83}$$

Fig. 6.1(a), 6.2(a) and 6.3(a) illustrate the singular value plot relative to the first residual generator. With blue solid line is indicated S_{f_1,r_1}, while the red dashed line indicates S_{f_2,r_1}. As it can be seen the nuisance f_2 is rejected for all the three different values of ρ. Fig. 6.1(b), 6.2(b) and 6.3(b) illustrate the singular value plot relative to the second residual generator. With blue solid line is indicated S_{f_2,r_2}, while the red dashed line indicates S_{f_1,r_2}. As it can be seen the nuisance f_1 is rejected for all the three different values of ρ. In order

Fig. 6.1(a), 6.2(a) and 6.3(a)	Fig. 6.1(b), 6.2(b) and 6.3(b)
blue solid line: $S_{f_1 r_1}(j\omega)$	blue solid line: $S_{f_2 r_2}(j\omega)$
red dashed line: $S_{f_2 r_1}(j\omega)$	red dashed line: $S_{f_1 r_2}(j\omega)$

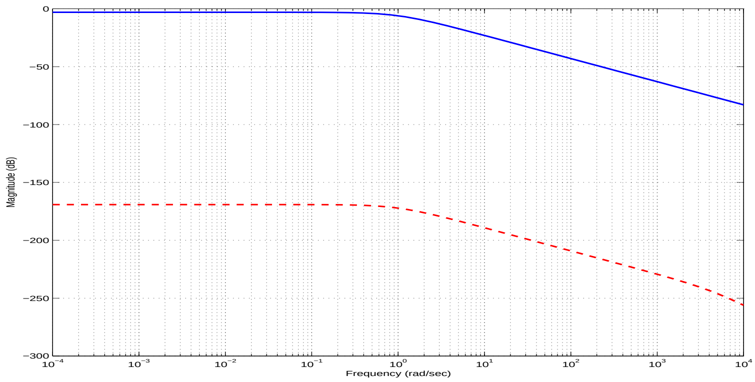
Table 6.1. Legend for Fig. 6.1, 6.2 and 6.3

to show the residual generator behavior for a time-varying parameter $\rho(t)$, a simulation is carried out. Define the parameter vector entries $\rho_1(t)$ and $\rho_2(t)$ by

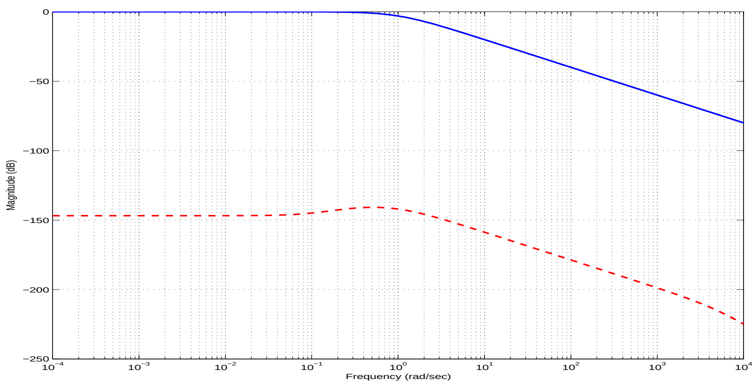
$$\rho_1(t) \doteq sin(0.62832t) \tag{6.84}$$

$$\rho_2(t) \doteq cos(0.3142t) \tag{6.85}$$

Fig. 6.5(b) illustrates the behavior of $\rho(t)$. Moreover the system is subject at $t = 30s$ to a sudden change in the first actuator and at $t = 60s$ in the second actuator, as depicted in Fig. 6.5(a). Fig. 6.4 illustrates the output of the residual generators. Notice that the two faults are overlapped for $70s$ and that the two residual generators reject efficiently the nuisance.

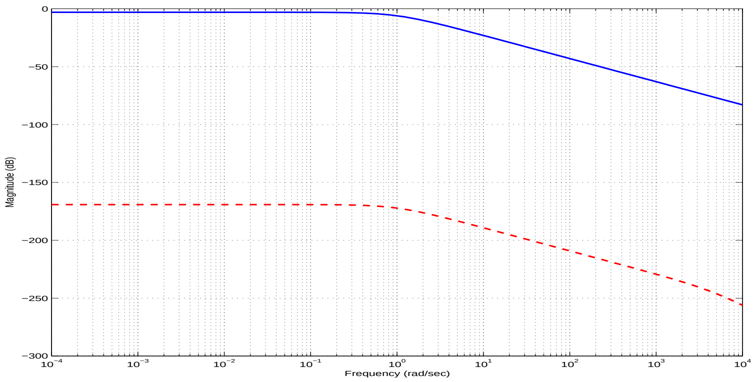


(a) The 1^{st} residual generator

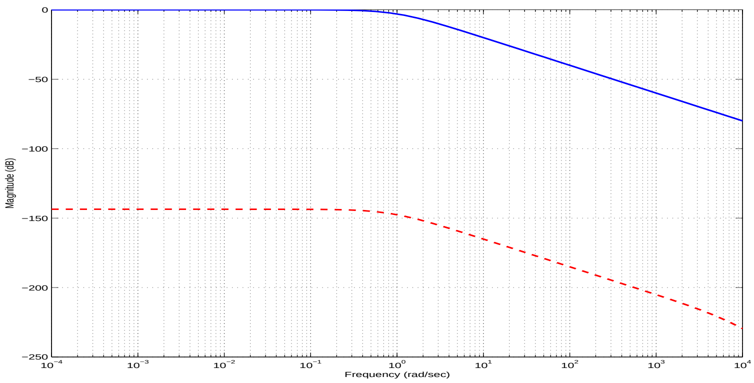


(b) The 2^{nd} residual generator

Figure 6.1. Singular value plot for $\rho = [0.5 \quad 0.2]'$

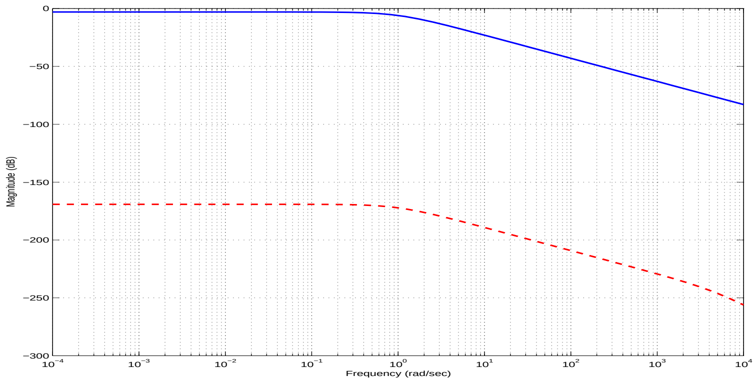


(a) The 1^{st} residual generator

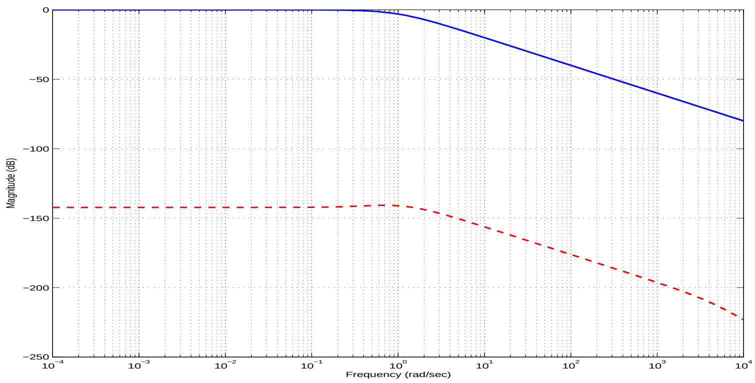


(b) The 2^{nd} residual generator

Figure 6.2. Singular value plot for $\rho = [-1 \quad -0.2]'$



(a) The 1^{st} residual generator



(b) The 2^{nd} residual generator

Figure 6.3. Singular value plot for $\rho = [1 \quad -0.4]'$

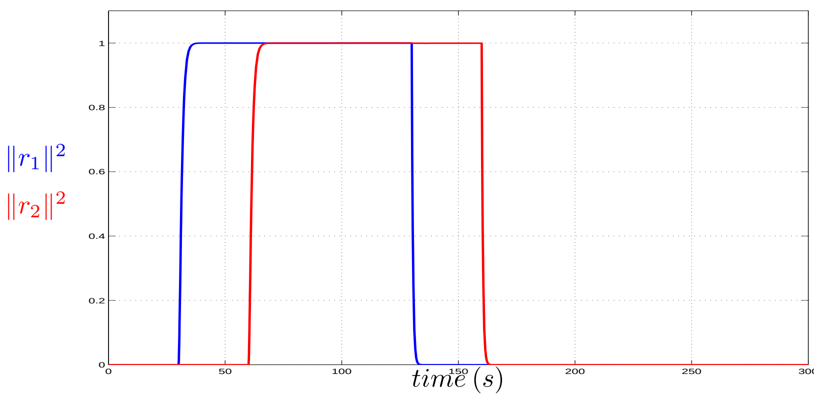
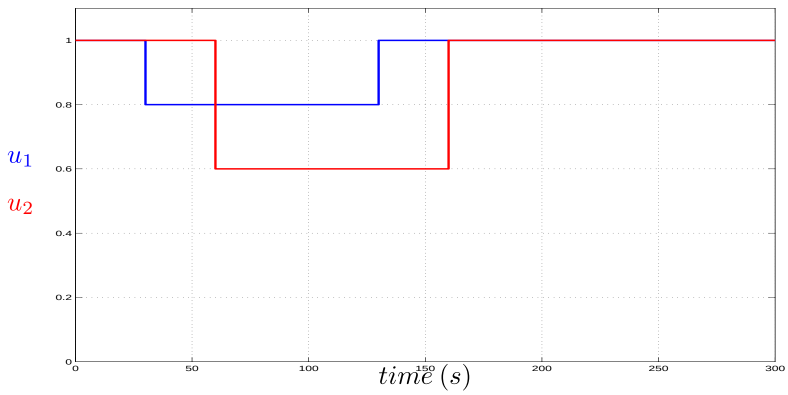
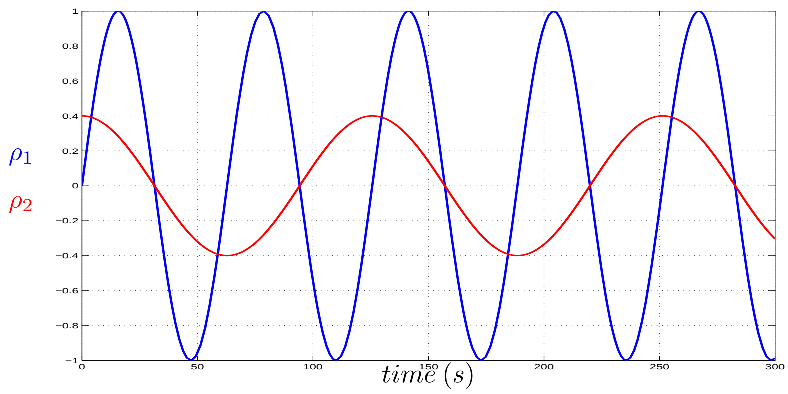


Figure 6.4. The residual generator outputs under faulty condition



(a) The actuator signals with faulty condition



(b) The ρ time-behavior

Figure 6.5. Simulation signals

6.6 Conclusions

This chapter has presented a solution to the FDI problem for ALPV system, which takes into account the sensitivity of the residual with respect to the fault. The main contribution is the introduction of a sensitivity constraint in the fault detection problem formulation and also the use of a cost functional to select the best observer. Nuisance attenuation is obtained through an $\mathcal{H}_\infty$ filtering design technique. Further work is needed to reduce some conservativeness, as explained in Remark 6.

Chapter 7

Results from comparative examples

7.1 The LTI case: a comparison example

This section explains the comparison between the results presented in Example 1 of [CS02] and the ones obtained using the method explained in chapter 4. Consider the following linear time-invariant system

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_\delta u_\delta(t) + B_{wg} u_{wg}(t) \\ y(t) &= Cx(t) \end{cases} \tag{7.1}$$

This system is a linearized model for the F16XL aircraft, with four states (longitudinal velocity x_u, normal velocity x_w, pitch rate x_q and pitch angle x_θ), one control input (u_δ elevon deflection angle) and an input disturbance (u_{wg} wind gust). All the four system states are assumed to be measurable. The system matrices are

$$A = \begin{bmatrix} -0.0674 & 0.0430 & -0.8886 & -0.5587 \\ 0.0205 & -1.4666 & 16.5800 & -0.0299 \\ 0.1377 & -1.6788 & -0.6819 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$B_\delta = \begin{bmatrix} -0.1672 \\ -1.5179 \\ -9.7842 \\ 0 \end{bmatrix}, \quad B_{wg} = \begin{bmatrix} 0.0430 \\ -1.4666 \\ -1.6788 \\ 0 \end{bmatrix}, \quad C = I_4, \tag{7.2}$$

It is assumed to have an actuator fault on u_δ and a fault on the sensor of the pitch angle x_θ state component, moreover the input disturbance u_{wg} is considered as a fault. The fault matrices F_θ, F_δ and F_{wg} are

$$F_\theta = \begin{bmatrix} 0 & -0.5587 \\ 0 & -0.0299 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F_\delta = B_\delta, \quad F_{wg} = B_{wg} \tag{7.3}$$

where F_θ represents the modeling of the y_θ sensor fault by a pseudo-actuator fault. In order to detect and isolate these faults, a bank of three residual generators is designed. Each residual generator is characterized by matrices L and H, which are designed with the method explained in chapter 4. Hence, each residual generator is like

$$\begin{cases} \dot{\hat{x}}(t) &= A\hat{x}(t) + B_\delta u_\delta(t) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= C\hat{x}(t) \\ r(t) &= H(y(t) - \hat{y}(t)) \end{cases} \tag{7.4}$$

Hence the following cost functional are defined

$$\mu_\delta \doteq \frac{\gamma_\delta^2 \lambda_\delta^2}{\|H_\delta C F_\delta\|^2} \tag{7.5}$$

$$\mu_{wg} \doteq \frac{\gamma_{wg}^2 \lambda_{wg}^2}{\|H_{wg} C F_{wg}\|^2} \tag{7.6}$$

$$\mu_\theta \doteq \frac{\gamma_\theta^2 \lambda_\theta^2}{\|H_\theta C F_\theta\|^2} \tag{7.7}$$

where the residuals satisfy the following inequalities

$$\|r_\delta\|_2^2 \leq \gamma_\delta^2(\|f_{wg}\|_2^2 + \|f_\theta\|_2^2) \quad (7.8)$$

$$\|r_{wg}\|_2^2 \leq \gamma_{wg}^2(\|f_\delta\|_2^2 + \|f_\theta\|_2^2) \quad (7.9)$$

$$\|r_\theta\|_2^2 \leq \gamma_\theta^2(\|f_\delta\|_2^2 + \|f_{wg}\|_2^2) \quad (7.10)$$

Fig. 7.1, 7.2 and 7.3 illustrate the behavior of the cost functionals μ_δ, μ_{wg} and μ_θ as function of λ_δ, λ_{wg} and λ_θ, respectively. In order to make comparable

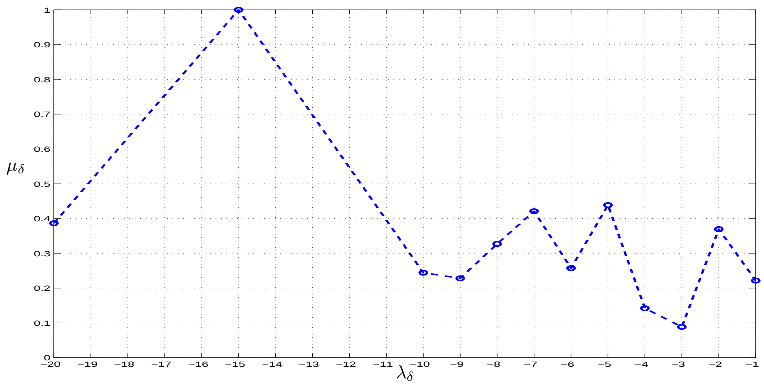


Figure 7.1. The μ_δ behavior

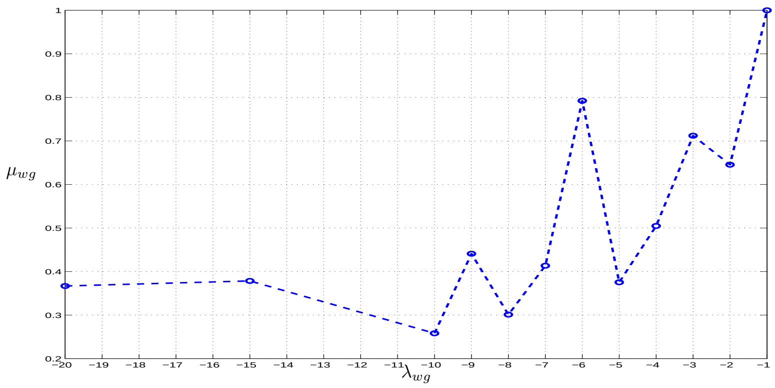


Figure 7.2. The μ_{wg} behavior

the results obtained with the proposed method with the ones from [CS02], a further constraint on the pole placing is introduced. Hence, Fig. 7.4 illustrates the singular value plot related to the first residual generator. The Fig. 7.5 and 7.6 show the singular value plot for the other faults.

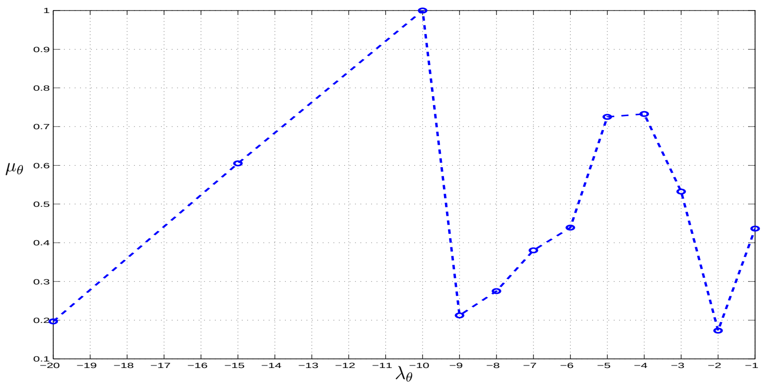
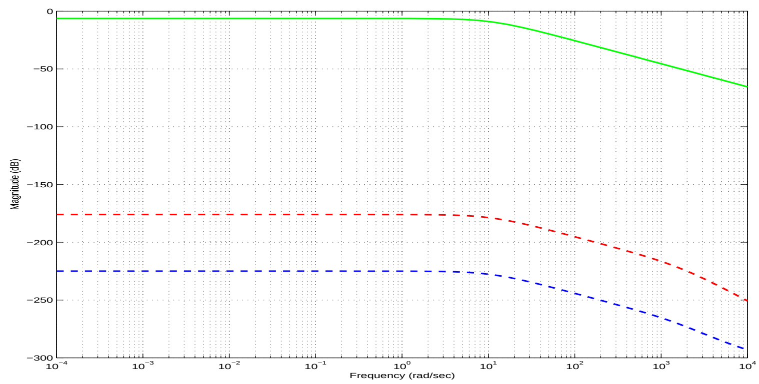


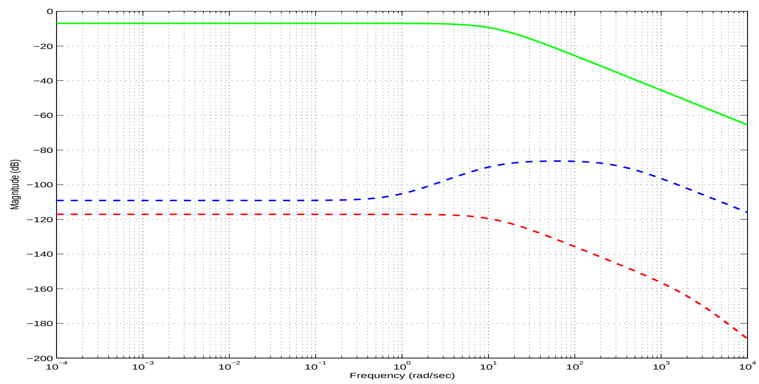
Figure 7.3. The μ_θ behavior

green solid line	$S_{f_\delta r_\delta}(j\omega)$	green dashed line	$S_{f_\delta\{r_{wg},r_\theta\}}(j\omega)$
blue solid line	$S_{f_{wg} r_{wg}}(j\omega)$	blue dashed line	$S_{f_{wg}\{r_\delta,r_\theta\}}(j\omega)$
red solid line	$S_{f_\theta r_\theta}(j\omega)$	red dashed line	$S_{f_\theta\{r_\delta,r_{wg}\}}(j\omega)$

Table 7.1. Legend for Fig. 7.4, 7.5 and 7.6

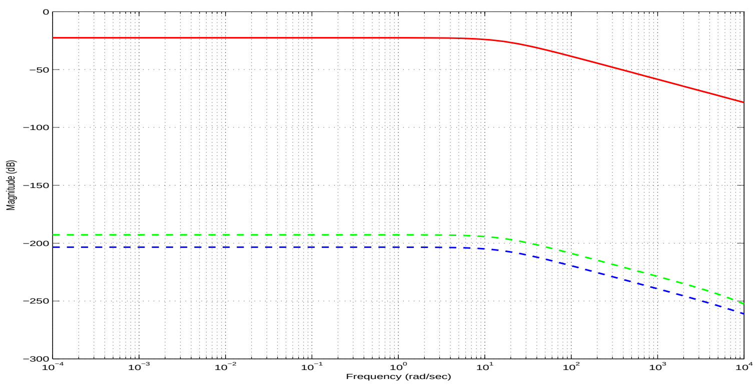


(a) The first residual generator : actuator fault, F_δ isolation

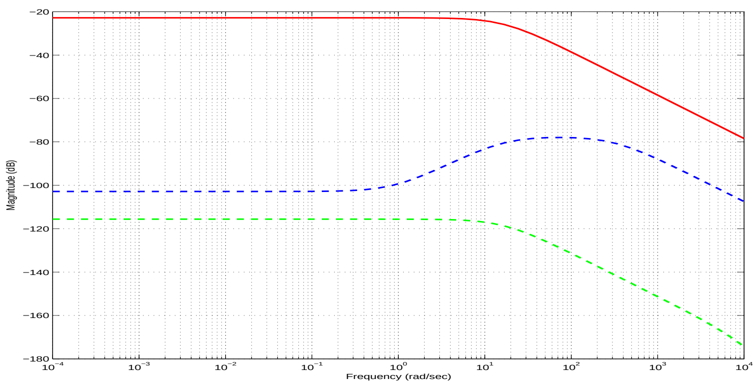


(b) [CS02]'s method: actuator fault, F_δ isolation

Figure 7.4. Singular value plot for the F_δ isolation

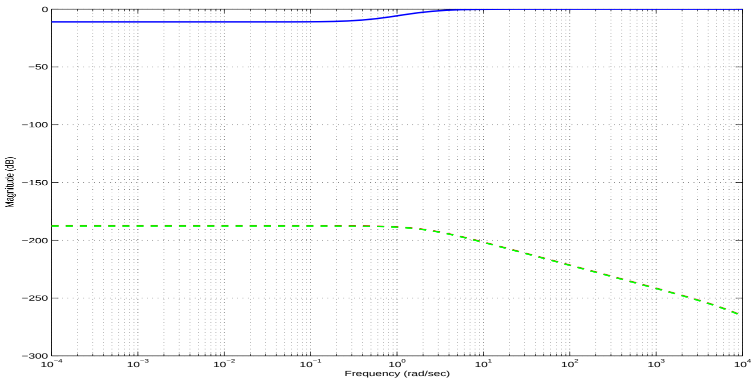


(a) The first observer: wind gust fault, F_{wg} isolation

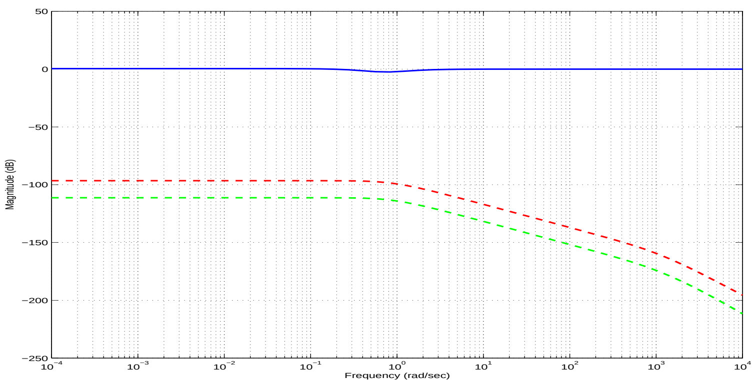


(b) [CS02]'s method: wind gust fault, F_{wg} isolation

Figure 7.5. Singular value plot for the F_{wg} isolation



(a) The first observer: sensor fault, F_θ isolation



(b) [CS02]'s method: sensor fault, F_θ isolation

Figure 7.6. Singular value plot for the F_θ isolation

7.1.1 Conclusions

The algorithm proposed for designing residual generator filters has shown better performance with respect to the method proposed in [CS02]. Moreover, the LMI optimization method allows to add the constraint on the filter pole placement in an easy way. Hence a stable filter with predetermined pole placement is designed.

7.2 The LPV case: a comparison example

The objective of this section is to use an FDI benchmark system to compare the results obtained through the ALPV FDI residual generator design explained in Chap. 6 with a nonlinear FDI residual generator design method. The bench-

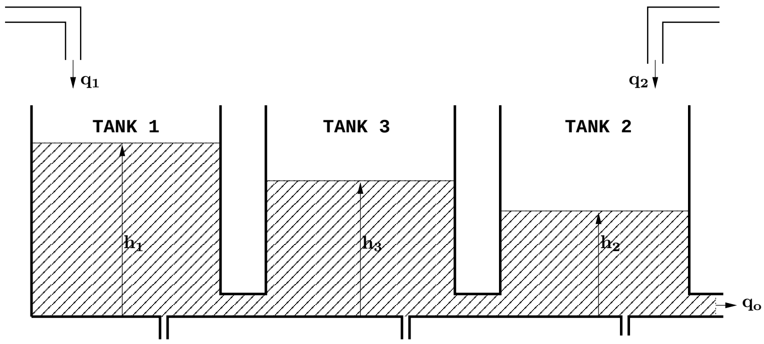


Figure 7.7. Three-Tank System

mark system is based on a three-tank system, [ZHM96] and [BK98]. The plant consists of three cylinders T_1, T_2 and T_3 with cross section A. These are connected serially by cylindrical pipes with a cross section S_n. Located at T_2 is the single so-called nominal outflow valve. The outflowing liquid is collected in a reservoir, which supplies the pumps 1 and 2. The three water levels h_1, h_2 and h_3 are available for control and fault diagnosis operations. It is assumed that the plant is subject to leakages in each cylinder, with cross section s_{l_1}, s_{l_2} and s_{l_3}. Table 7.2 shows the numerical values of the physical parameters of the system. The mathematical model is obtained through the Torricelli rule, then the three levels h_1, h_2 and h_3 are subject to the following relations

$$\begin{cases} A\dot{h}_1 = & q_1 - \mathbf{s}(h_1 - h_3)S_n\sqrt{2g|h_1 - h_3|} - s_{l_1}\sqrt{2gh_1} \\ A\dot{h}_2 = & q_2 + \mathbf{s}(h_3 - h_2)S_n\sqrt{2g|h_3 - h_2|} - S_n\sqrt{2gh_2} - s_{l_2}\sqrt{2gh_2} \\ A\dot{h}_3 = & \mathbf{s}(h_1 - h_3)S_n\sqrt{2g|h_1 - h_3|} - \mathbf{s}(h_3 - h_2)S_n\sqrt{2g|h_3 - h_2|} - s_{l_3}\sqrt{2gh_3} \end{cases} \tag{7.11}$$

With q_1, q_2 is indicated the incoming mass flow, while $\mathbf{s}(\cdot)$ indicates the sign function, i.e.

$$\mathbf{s}(x) \doteq \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \tag{7.12}$$

In order to proceed with the design of the ALPV residual generator filters, a

Parameter	Value
A	$0.0154(m^2)$
S_n	$5 \times 10^{-5}(m^2)$

Table 7.2. Three-tank parameter values

quasi-LPV model of (7.11) is introduced

$$\begin{cases} A\dot{h}(t) &= A(h)h(t) + Bq(t) + \sum_{i=1}^{3} F_i f_i(t) \\ y(t) &= Ch(t) \end{cases} \tag{7.13}$$

where

$$A(h) = \begin{bmatrix} -\frac{S_n\sqrt{2g}}{\sqrt{|h_1 - h_3|}} & 0 & \frac{S_n\sqrt{2g}}{\sqrt{|h_1 - h_3|}} \\ 0 & -\frac{S_n\sqrt{2g}}{\sqrt{|h_3 - h_2|}} - \frac{S_n\sqrt{2g}}{\sqrt{h_2}} & \frac{S_n\sqrt{2g}}{\sqrt{|h_3 - h_2|}} \\ \frac{S_n\sqrt{2g}}{\sqrt{|h_1 - h_3|}} & \frac{S_n\sqrt{2g}}{\sqrt{|h_3 - h_2|}} & -\frac{S_n\sqrt{2g}}{\sqrt{|h_1 - h_3|}} - \frac{S_n\sqrt{2g}}{\sqrt{|h_3 - h_2|}} \end{bmatrix} \tag{7.14}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = I_3 \quad F_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad F_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7.15}$$

The fault signals f_i, $i = 1, 2, 3$ are defined by

$$f_1(t) = \beta(t - t_{f_1}) s_{l_1} \sqrt{2gh_1} \tag{7.16}$$

$$f_2(t) = \beta(t - t_{f_2}) s_{l_1} \sqrt{2gh_2} \tag{7.17}$$

$$f_3(t) = \beta(t - t_{f_3}) s_{l_1} \sqrt{2gh_3} \tag{7.18}$$

where each $\beta(t - t_{f_i})$ is defined as in (1.4). In practice the leakage is modeled as a circular hole that increases its radius until a maximum is reached. In order to define an ALPV model for the three tank system, the parameter vector $\rho(t)$ can be defined as follows

$$\rho(t) \doteq \begin{bmatrix} \rho_1(t) \\ \rho_2(t) \\ \rho_3(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{|h_1 - h_3|}} \\ \frac{1}{\sqrt{|h_3 - h_2|}} \\ \frac{1}{\sqrt{h_2}} \end{bmatrix} \tag{7.19}$$

Then $A(\rho)$ is defined by

$$A(\rho) \doteq A_1\rho_1(t) + A_2\rho_2(t) + A_3\rho_3(t) \tag{7.20}$$

with

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -S_n\sqrt{2g} & 0 & S_n\sqrt{2g} \\ 0 & 0 & 0 \\ S_n\sqrt{2g} & 0 & -S_n\sqrt{2g} \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S_n\sqrt{2g} & S_n\sqrt{2g} \\ 0 & S_n\sqrt{2g} & -S_n\sqrt{2g} \end{bmatrix} \\
A_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S_n\sqrt{2g} & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{7.21}
$$

Notice that the system (7.13) is well defined for $h_2(t) \neq 0$, $h_1(t) \neq h_3(t)$ and $h_2(t) \neq h_3(t)$. This implies that the operating levels maintain the initial order, i.e. if $h_1 > h_3 > h_2$ this relation must be satisfied for all the simulation times. The input mass flow is controlled by a PI-controller, in such a way the first an the second tank liquid level can track a predefined level. The control law is defined by

$$q_1(t) = k_p(h_1^{ref} - h_1(t)) + k_i \int (h_1^{ref} - h_1(t))dt \tag{7.22}$$

$$q_2(t) = k_p(h_2^{ref} - h_2(t)) + k_i \int (h_2^{ref} - h_1(t))dt \tag{7.23}$$

Notice that our main objective is the FDI of the plant. The ALPV residual generator for each i-th fault is defined by

$$
\begin{cases}
\dot{\hat{x}}(t) = A(\rho)\hat{x}(t) + B(\rho)q(t) + L_i(\rho)(y(t) - C\hat{x}(t)) \\
\hat{y}(t) = C\hat{x}(t) \\
r_i(t) = H_i\,(y(t) - \hat{y}(t))
\end{cases} \tag{7.24}
$$

with

$$L_i(\rho(t)) = L_{i1}\rho_1(t) + L_{i2}\rho_2(t) + L_{i3}\rho_3(t) \tag{7.25}$$

where the matrices L_{i1}, L_{i2}, L_{i3} and H_i are obtained as solution of the optimization problem (6.40). The residuals r_i, $i = 1, 2, 3$, satisfies the following relations

$$\|r_1\|_2^2 \leq \gamma_1^2(\|f_2\|_2^2 + \|f_3\|_2^2) \tag{7.26}$$

$$\|r_2\|_2^2 \leq \gamma_2^2(\|f_1\|_2^2 + \|f_3\|_2^2) \tag{7.27}$$

$$\|r_3\|_2^2 \leq \gamma_3^2(\|f_1\|_2^2 + \|f_2\|_2^2) \tag{7.28}$$

The nonlinear observer are obtained as explained in [PI01] and [PI00]. Then, the nonlinear residual generator are defined by

$$\begin{cases} A\dot{\xi}_1 = q_1 - \mathbf{s}(h_1 - h_3)S_n\sqrt{2g|h_1 - h_3|} + k_1(h_1 - \xi_1) \\ r_1 = h_1 - \xi_1 \end{cases} \tag{7.29}$$

$$\begin{cases} A\dot{\xi}_2 = q_2 + \mathbf{s}(h_3 - h_2)S_n\sqrt{2g|h_3 - h_2|} - S_n\sqrt{2gh_2} + k_2(h_2 - \xi_2) \\ r_2 = h_2 - \xi_2 \end{cases} \tag{7.30}$$

$$\begin{cases} A\dot{\xi}_3 = \mathbf{s}(h_1 - h_3)S_n\sqrt{|h_1 - h_3|} - \mathbf{s}(h_3 - h_2)S_n\sqrt{2g|h_3 - h_2|} + k_3(h_3 - \xi_3) \\ r_3 = h_3 - \xi_3 \end{cases} \tag{7.31}$$

where k_1, k_2 and k_3 indicate, respectively, the observer gains, which are chosen positive for stability reasons. Notice that, the structure of the three tank system is such that each fault can be decoupled efficiently from the others. In conclusion, with the nonlinear method three reduced-order nonlinear observer are designed, each one capable to detect and isolate the respective faults. While, with the ALPV design method three full-order observer are obtained. In order to choose a proper value of λ for each ALPV residual generator a computation of μ_i, $i = 1, 2, 3$, is done.

$$\mu_1 = \frac{\gamma_1^2 \lambda_1^2}{\|H_1 C F_1\|^2} \tag{7.32}$$

$$\mu_2 = \frac{\gamma_2^2 \lambda_2^2}{\|H_2 C F_2\|^2} \tag{7.33}$$

$$\mu_3 = \frac{\gamma_3^2 \lambda_3^2}{\|H_3 C F_3\|^2} \tag{7.34}$$

Fig. 7.8 shows the behavior of each μ_i for a grid of λ values. One can note that the behavior of μ_1 and μ_2 are similar, this can be explained by the fact that the dynamics of h_1 and h_2 are similar. Notice that for smaller values of λ the values of the ratios μ_i are decreasing, then for smaller values of λ one has filters with better fault sensitivity. On the contrary smaller values of λ indicate filters with slower time response from the fault signal to the residual. Hence, the choice for the proper value of λ is based on the main filter property that is desired, i.e. for better fault sensitivity one has to choose filters with slower time response. In order to compare the two distinct residual generator design methods, a simulation is carried out, based on the nonlinear model of the three-tank system. To this end, table 7.3 show the observer gains used in the

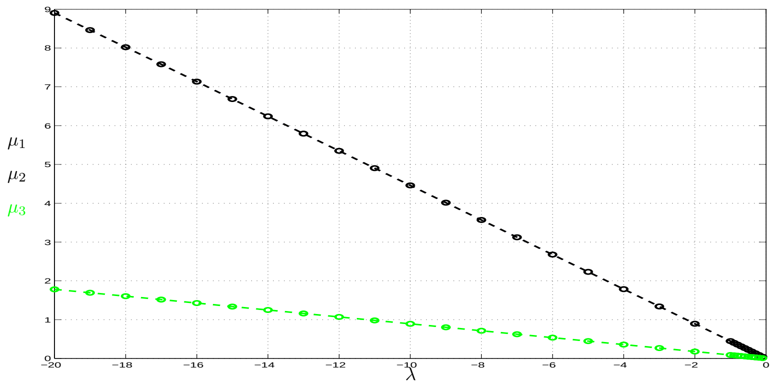
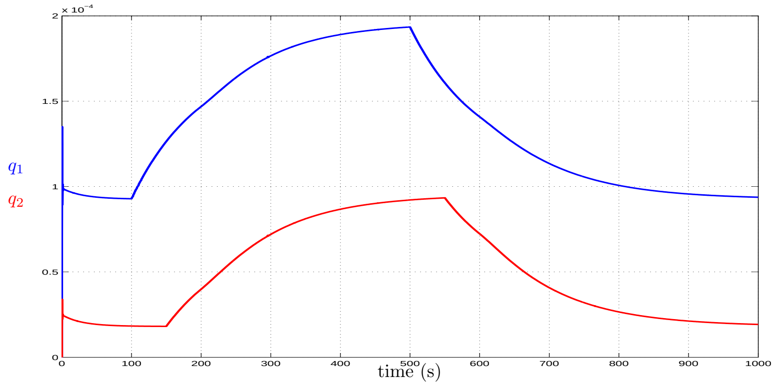


Figure 7.8. ratio

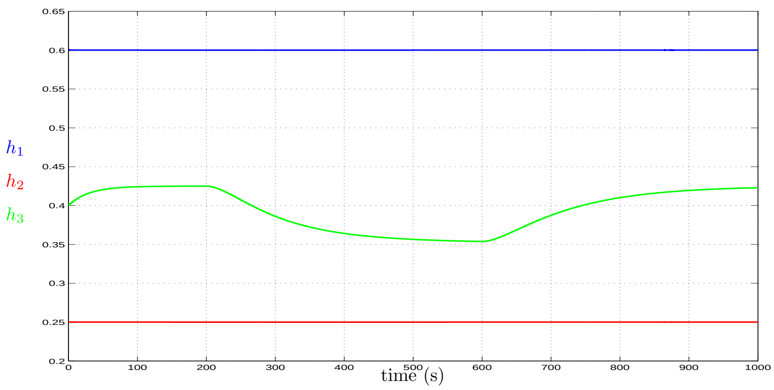
Parameter	Value	Parameter	Value
k_1	1	λ_1	-1
k_2	1	λ_2	-1
k_3	1	λ_3	-1

Table 7.3. Residual generator parameters

simulation. Two simulations are carried out, with different fault time-behavior, the first is executed with the assumption that the faults are incipient, whilst in the second the faults have an abrupt time-behavior. The first simulation involves a leakage in each tank, each leakage is simulated by a circular hole with increasing radius from zero to a predetermined maximum value. Moreover, the three leakages are overlapped, in order to verify the isolation property of the filters. It is assumed that the leakage in the first tank starts at $100s$ of the simulation time, while the leakage in the second tank starts at $150s$ and the leakage in the third tank at $200s$, as shown in Fig. 7.10(b). Fig. 7.9(a), 7.9(b) and 7.10(a) show the input action on the system, the system's liquid levels and the behavior of the time-varying parameter ρ, respectively. Fig. 7.11 shows the residual generator filters time response. Notice that, the nonlinear and the ALPV residual generators have a similar dynamics, in fact the dominant pole is equal to -1 for both filters. Hence, the time responses of the nonlinear and ALPV filters are similar. Whilst, the ALPV residual generator related to the leakage in the third tank has a faster response with respect to the nonlinear filter. Fig. 7.11(a), 7.11(b) and 7.11(c) illustrate the residual generator time response from the fault signal to the residual. Another simulation is carried out with the assumption that the leakage are instantaneous, i.e. the leakage is simulated as a circular hole with a predetermined radius. The residual generators are the same that are used in the previous simulation. Fig. 7.13(b). Fig. 7.12(a), 7.12(b) and 7.13(a) show the input action on the system, the system's liquid levels and the behavior of the time-varying parameter ρ, respectively. Fig. 7.14(a), 7.14(b) and 7.14(c) illustrate the residual generator time response from the fault signal to the residual.

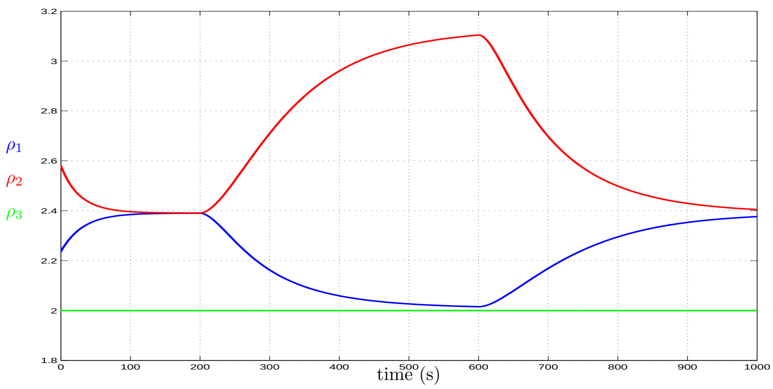


(a)

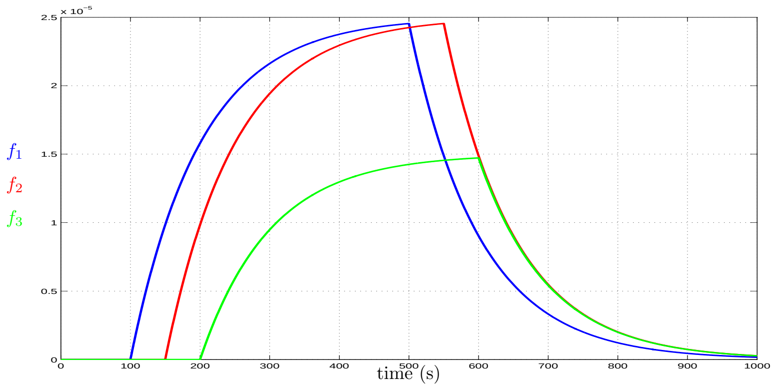


(b)

Figure 7.9. incipient

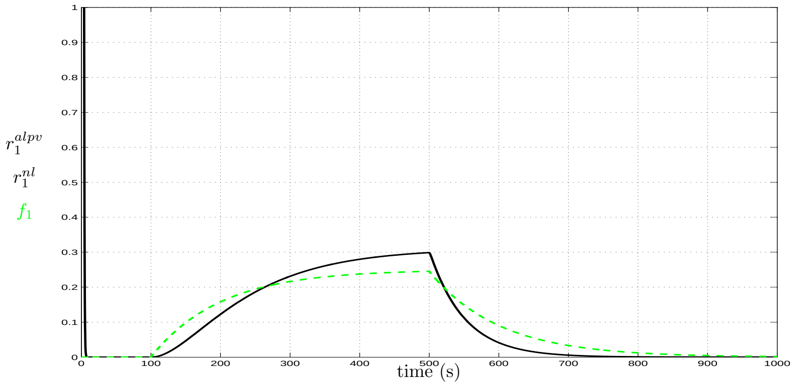


(a)

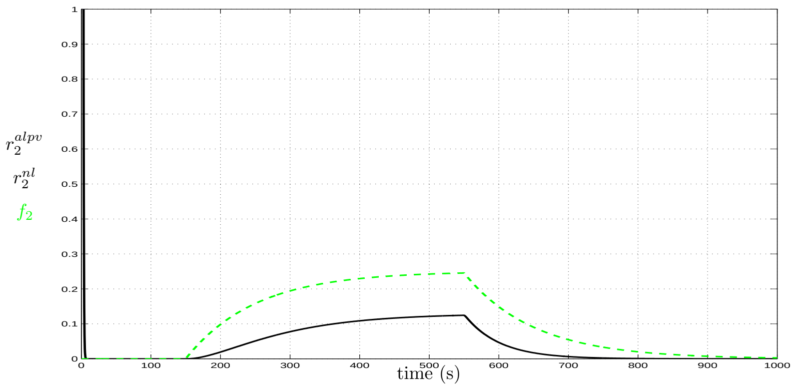


(b)

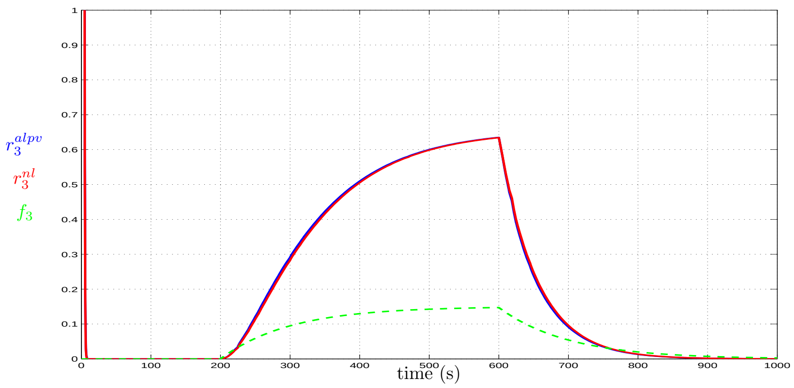
Figure 7.10. incipient



(a)

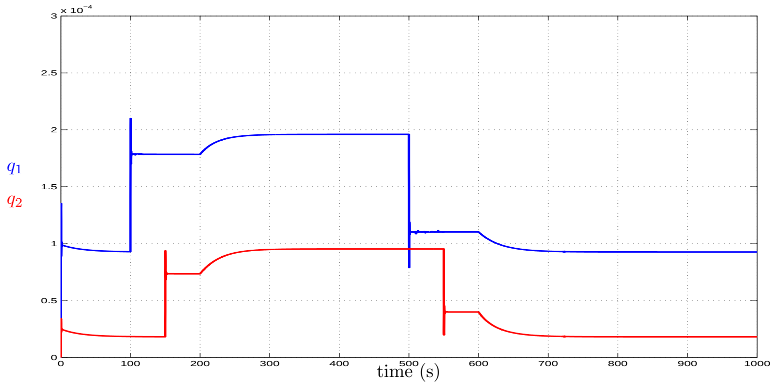


(b)

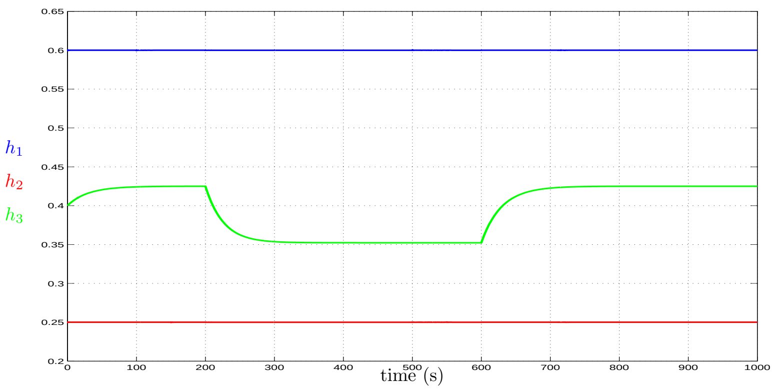


(c)

Figure 7.11. incipient

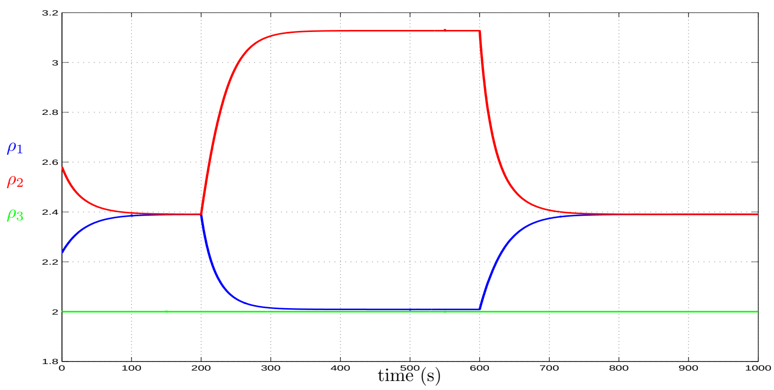


(a)

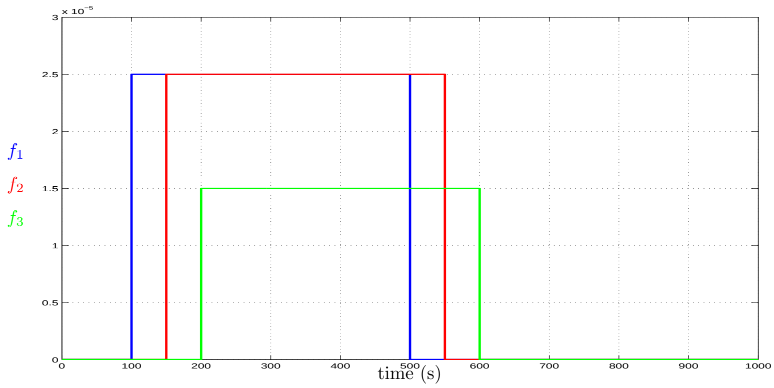


(b)

Figure 7.12. abrupt

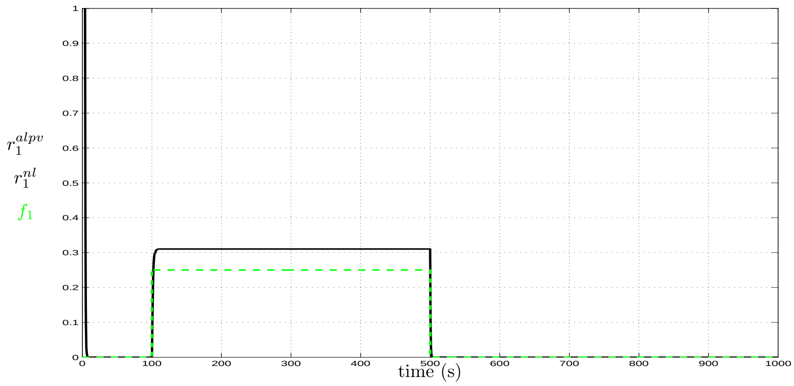


(a)

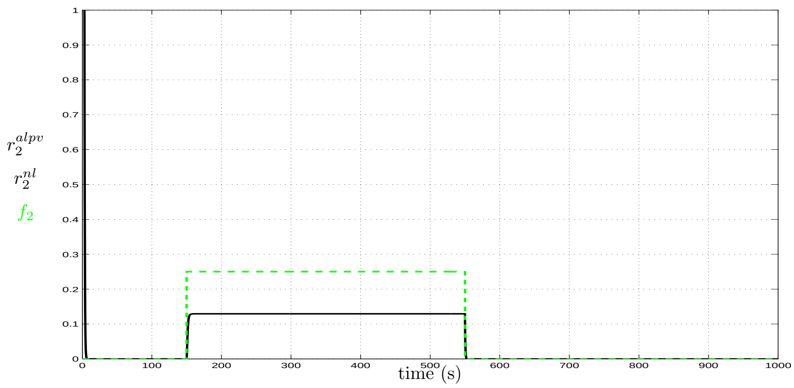


(b)

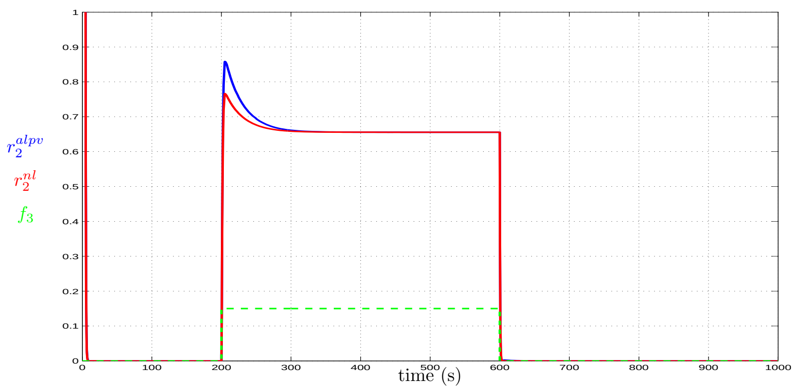
Figure 7.13. abrupt



(a)



(b)



(c)

Figure 7.14. Abrupt fault

7.2.1 Conclusions

The three tank system explains the capability of the ALPV residual generator to be used in a nonlinear environment. In fact LPV based methods can be considered as an extension of gain scheduling control for nonlinear systems. The ALPV design method is based on an optimization problem which can set the filter dynamics in such a way the filter has the maximum fault sensitivity and the minimum $\mathcal{H}_\infty$ gain from nuisances to residual. Moreover, the LMI optimization method allows to add the constraint on the filter pole placement in an easy way. Hence a stable filter with predetermined pole placement is designed. The simulations show that the ALPV residual generator can be used in a nonlinear environment, with the same performance of a nonlinear residual generator filter.

Conclusion

This thesis has introduced a new algorithm for designing a fault detection and isolation (FDI) filter with enhanced fault detection capability. This algorithm is derived from a bilinear matrix inequality (BMI) optimization problem. Then, the BMI optimization problem is solved through a family of linear matrix inequality (LMI) optimization problems. Chapter 4 has introduced the FDI algorithm for linear time-invariant (LTI) systems subject to disturbances. The filter design method leads to the unknown-input observer. In particular the problem of enhancing the fault detection capability of the filter is addressed through an eigen-structure assignment. With this geometric constraint it is possible to map the fault injection in a predetermined fixed residual subspace. While the disturbances attenuation is solved through an optimal $\mathcal{H}_\infty$ filtering problem. Furthermore, some assumptions are given in order to make the FDI problem well-posed. Hence, if the LTI system satisfies these assumptions the LMI optimization problem has a solution, then an FDI filter exists. The chapter is ended by a practical design example, which compares the results from two filters, one designed with the eigen-structure constraint and the other by taking into account the disturbance attenuation problem only. In such a way it is possible to appreciate the benefits introduced by the eigen-structure assignment constraint. Chapter 5 has introduced the study of FDI residual generator design problem in systems with model uncertainties. The filter design method leads to the unknown-input observer. In particular the problem of enhancing the fault detection capability of the filter is addressed through an eigen-structure assignment. While the disturbances attenuation is solved through an optimal $\mathcal{H}_\infty$ filtering problem. Some assumptions are given in order to make the FDI problem well-posed. In conclusion a stable residual generator filter irrespective on the model uncertainties is obtained. The chapter is ended by a practical design example, in which are shown the filter fault detection capabilities and its robustness to model uncertainties. Chapter 6 has introduced the study of fault detection in linear parametric variable (LPV) system. The LPV modeling techniques are particularly appealing because nonlinear plants can be treated as linear systems with a priori not necessarily known but on-line measurable time varying parameters. Moreover, LPV based methods can be considered as an extension of gain scheduling control for nonlinear systems. The residual generator filters are synthesized by robust filtering design methods under an eigen-structure assignment constraint, in order to enhance the fault detection capability of the filter. The design method leads to the unknown-input observer. While, the disturbances attenuation is solved through an optimal $\mathcal{H}_\infty$ filtering problem. With this geometric constraint it is possible to map the fault injection in a predetermined fixed residual subspace. Moreover, a ratio between the zero frequency gain from fault to residual and the $\mathcal{H}_\infty$ gain from disturbances to residual is defined as performance criterion. Then, the minimization of this ratio leads to a residual generator with the maximum fault transmission and the minimum disturbance transmission to the residual. In conclusion the residual generator is obtained as a solution of a family of LMI optimization problems. Some assumptions are given in order to make the filter design problem well-posed. Then, if the LPV system satisfies these assumptions a stable LPV FDI filter exists. A practical designing example ends the chapter. Chapter 7 has shown the results from the comparison between residual generator filters designed with the methods proposed in this thesis and other designed with pre-existing methods. In particular, the LTI design method is compared with the method exposed in [CS02]. From comparisons it emerges that the proposed method features better fault detection capabilities. The LPV design method is compared with a nonlinear design method, both applied to a benchmark system. In conclusion the ALPV residual generator has shown a behavior similar to the nonlinear filter.

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