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Some properties of pattern avoiding permutations

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Chapter 1

Introduction

The study of pattern avoiding permutations can be directly related to D. Knuth who introduced it in the 1960's, posing a stack sorting problem [Kn]. In the successive period pattern avoiding permutations have always been sparking several authors' interest, but only in the 1980'-90's this matter has come to the fore, catching many attentions.

The Stanley-Wilf conjecture (*for any pattern q, there exists a constant c_q so that for all positive integers, we have $S_n(q) \leq c_q^n$*) dates back approximately twenty years ago. A lot of papers are about this long-standing conjecture: R. Arratia's study on an equivalent reformulation of it (*the limit $\lim_{n\to\infty} \sqrt[n]{S_n(q)}$ does exist* [Ar]); its proofs for layered patterns [B1] and for patterns of length four [B]; Klazar's paper on its link with the Füredi-Hajnal conjecture [FH] involving $0-1$ matrices [Kl]; its solution due to A. Marcus and G. Tardos [MT] (for more details on the Stanley-Wilf conjecture we refer to [B2]).

At the beginning of the 1990's, the studies of J. West [W1, W3] on permutations sortable by two passages through a stack appeared, directly connected to the stack-sorting operation posed by D. Knuth [Kn]. These permutations were characterized in terms of pattern avoiding permutations (more precisely they belong to $S_n(2341, 3\bar{5}241)$) and their cardinality is

$2(3n)!/(n+1)!(2n+1)!$. This result, first conjectured by J. West himself, was then proved by D. Zeilberger [Z] with analytic techniques; and even that has induced a series of interesting works, since $2(3n)!/(n+1)!(2n+1)!$ is also the cardinality of a certain family of planar maps (rooted nonseparable maps). Starting from this correspondence S. Dulucq, S. Gire and J. West [DGW] found a class of pattern avoiding permutations ($S_n(2413, 41\bar{3}52)$) also enumerated by $2(3n)!/(n+1)!(2n+1)!$. This class, by means of bijective passages, was together mapped to $S_n(2341, 3\bar{5}241)$ by S. Dulucq, S. Gire and O. Guibert [DGG], providing a combinatorial proof of J. West's conjecture and an enumeration of two-stack sortable permutations according to various parameters.

We can not forget to mention the paper authored by R. Simion and F. W. Schmidt [SS] which represent the first methodical work on the matter: here, the enumeration of permutations avoiding any subset of patterns of length three is solved; even though it has been preceded by similar works, it is the first where such a problem is tackled in an exhaustive way.

The above examples are only an arbitrary choice among all the possible ones to show how the matter of pattern avoidance is an interesting and relevant discipline in Mathematics. The list of subjects where pattern avoiding permutations are involved in could be very long. Here, only a few of them are cited: sorting problems [BoM, Kn, Rot, W1, W3], analysis of regularities in words [Be, L], particular instances of pattern matching algorithms optimizations [BBL]... Nevertheless, we recall that pattern avoiding permutations are proved to be useful also in other disciplines not belonging to Computer Science. For instance, they arise in the study of the singularities of Schubert varieties [LS], Chebyshev polynomials of the second kind [Kra, MV1], rook polynomials for a rectangular board [MV]. Moreover, the 2143-avoiding permutations, called vexillary permutations, are used in the theory of Schubert polynomials. Many other papers dealing with pattern avoidance have been published and it would not be possible to cite all of them.

Beside the enumeration of permutations avoiding one or more patterns, many variations on this main theme have been introduced and studied by several authors. We recall the enumeration of permutations containing a certain pattern [MV2, No, NZ], counting the occurrences of a certain pattern in permutations, counting the permutations avoiding patterns with different lengths [M1] and, last but not least, studying permutations avoiding patterns with increasing length. The paper of A. Regev [R] can also be referred to the latter, where the author gives an asymptotic value of the number of permutations avoiding the subsequences $12\dots k$. By the way, using the results of this work, it is possible to prove the Stanley-Wilf conjecture for the permutations of $S_n(12\dots k)$. T. Mansour, too, in the great amount of his coauthored works, was concerned with patterns of increasing length: in [M2], for instance, he provides a simple expression for the number of permutation avoiding the sequences of length k having the first entry equal to a certain value (but this is only one among his many results!). Close to this line of research is the paper [BDPP3] where the authors count the permutations avoiding an increasing number of length-increasing patterns. The same authors in [BDPP2] characterized the permutations avoiding the pattern 321 and a certain increasing length pattern: their enumeration provided a kind of "discrete continuity" between Motzkin and Catalan numbers. Chapter 3 of the present thesis presents similar results, providing a continuity between Fibonacci and Catalan numbers basing on pattern avoiding permutations, where the forbidden patterns are suitably generalized considering them more and more longer, till they are no more relevant with respect to their occurrence in the permutation.

Only recently, Babson and Steingrímsson introduced a particular class of forbidden patterns, namely the generalized permutation patterns [BS]. A generalized pattern τ is a permutation equipped with some dashes between some pairs of its element (e.g., $1-32$ and $2-43-1$ are generalized patterns of length 3 and 4, respectively) and a permutation contains τ when adjacent elements in τ correspond to adjacent elements in the permutation. The authors introduced these kind of patterns for the study of Mahonian statistics in permutations. They proved that almost all known Mahonian permutation statistics can be written as linear combination of generalized patterns of length at most 3. Successively, several classes of generalized pattern avoiding permutations have been widely studied in recent years. We only cite [K], where the author presents a wide set of interesting questions about the matter, including some aspects of the enumeration of permutations avoiding certain sets of generalized patterns with some restrictions.

The enumerations of permutations avoiding one, two or three Babson-Steingrímsson patterns were already tackled and solved in [C], [CM] and [BFP], respectively. Chapter 2 is devoted to the enumeration of the permutations avoiding more than three Babson-Steingrímsson patterns (generalized patterns of length three), this is the reason why it can be seen as the continuation of the work started in [BFP] for the fulfillment of the proofs of the conjectures presented in [CM]. Moreover, in the same chapter we enumerate the permutations avoiding one generalized pattern of length three according to the length of the permutations and its last or first entry.

The matter of generalized pattern avoiding permutations has received a further attention in the paper [El] where the author, motivated by the recent proof of the Stanley-Wilf conjecture, investigates about the behavior of the number of permutations avoiding a generalized pattern.

An additional aspect which can be introduced in the analysis of pattern avoiding permutations relates to their order properties. In Chapter 4 some classes of pattern avoiding permutations are studied under this point of view.

Thanks to some bijections with other combinatorial objects (Dyck paths, Motzkin paths and Schröder paths) it is possible to transfer to some class of permutations a natural order defined on them [FP2]. This order is such that the mentioned paths are endowed with a distributive lattice structure. We achieve that, in same case, the induced partial order on the obtained subsets of restricted permutations coincides with the strong Bruhat order of the symmetric group S_n, so that they can be regarded as distributive sublattices of S_n (which is not a lattice if considered as a whole). It can be noted that similar results have been found by other authors [BW, Dr], nevertheless they were concerned with the weak order on permutations.

The last chapter of the thesis deals with some considerations about the exhaustive generation of combinatorial objects. More precisely, we outline a procedure to generate all the objects of a class such that a Gray code is obtained. This possibility is connected with a particular property of the succession rule encoding the construction of the objects, which is not too much unusual among the different kinds of succession rules. Moreover, an efficient generation algorithm for Dyck paths of the same length is proposed. The main idea which it is based on can be easily extended to Grand Dyck and Motzkin paths, nevertheless we think that some suitable consideration of the same kind can be find also for some class of pattern avoiding permutations.

1.1 Background

Pattern and generalized pattern avoiding permutations

A (classical) *pattern* is a permutation $\sigma \in S_k$ and a permutation $\pi \in S_n$ *avoids* σ if there is no any subsequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ which is order-isomorphic to σ. In other word, π must contain no subsequences having the entries in the same relative order of the entries of σ. *Generalized patterns* were introduced by Babson and Steingrímsson for the study of the mahonian statistics on permutations [BS]. They are constructed by inserting one or more dashes among the elements of a classical pattern (two or more consecutive dashes are not allowed). For instance, $216-4-53$ is a generalized pattern of length 6. The *type* $(t_1, t_2, \ldots, t_{h+1})$ of a generalized pattern containing h dashes records the number of elements between two dashes (we suppose a dash at the beginning and at the end of the generalized pattern, but we omit it): the type of $216-4-53$ is $(3,1,2)$. A permutation π *contains* a generalized pattern τ if π contains τ in the classical sense and if any pair of elements of π corresponding to two adjacent elements of τ (not separated by a dash) are adjacent in π, too. For instance, $\pi = 153426$ contains $32-14$ in the entries $\pi_2\pi_3\pi_5\pi_6 = 5326$ or the pattern $3-214$ in the entries $\pi_2\pi_4\pi_5\pi_6 = 5426$. A permutation π *avoids* a generalized pattern τ if it does not contain τ. If P is a set of generalized patterns, we denote $S_n(P)$ the permutations of length n of S (symmetric group) avoiding the patterns of P.

Here, we are interested to the generalized patterns of length three, which are of type $(1,2)$ or $(2,1)$ and are those ones specified in the set

$$\begin{aligned}\mathcal{M} = \ & \{1-23, 12-3, 1-32, 13-2, 3-12, 31-2, 2-13, 21-3,\\ & 2-31, 23-1, 3-21, 32-1\}.\end{aligned}$$

In the sequel, sometimes we can refer to a *generalized pattern of length three* more concisely with *pattern*.

If $\pi \in S_n$, we define its *reverse* and its *complement* to be the permutations π^r and π^c, respectively, such that $\pi_i^r = \pi_{n+1-i}$ and $\pi_i^c = n+1-\pi_i$. We generalize this definition to a generalized pattern τ obtaining its reverse τ^r by reading τ from right to left (regarding the dashes as particular entries) and its complement τ^c by considering the complement of τ regardless of the dashes which are left unchanged (e.g. if $\tau = 216-4-53$, then $\tau^r = 35-4-612$ and $\tau^c = 561-3-24$). It is easy to check

$\tau^{rc} = \tau^{cr}$. If $P \subseteq \mathcal{M}$, the set $\{P, P^r, P^c, P^{rc}\}$ is called the *symmetry class* of P (P^r, P^c and P^{rc} contain the reverses, the complements and the reverse-complements of the patterns specified in P, respectively). We have that $|S_n(P)|{=}|S_n(P^r)|{=}|S_n(P^c)|{=}|S_n(P^{rc})|$ (see [SS]), therefore we can choose one of the four possible sets as the *representative* of a symmetry class, as far as the enumeration of $S(R),\ R \in \{P, P^r, P^c, P^{rc}\}$, is concerned.

Eco method and succession rules

Often, we are going to refer to the ECO method, basing our arguments on the ECO construction of some combinatorial objects. This method allows to construct all the objects of a given class. If p is a parameter according to which we enumerate the objects, the ECO method is based on the possibility to define an operator generating all the objects of size $n+1$ (i.e., the objects whose parameter has value $n+1$) exactly ones starting from the objects of size n. So we have a recursive description of the objects we can often encode with a *succession rule* (see below), from which, in many cases, it is possible to derive the generating function of the class. Here, we present only the main theorem the ECO method relies on, for more details see [BDPP1].

Theorem 1.1.1 *Let S be a class of combinatorial objects; let p be a parameter of S ($p : S \to \mathbb{N}^+$) and $S_n = \{x \in S : p(x) = n\}$; let ϑ be an operator on S ($\vartheta : S_n \to 2^{S_{n+1}}$, where $2^{S_{n+1}}$ is the power set of S_{n+1}). If ϑ satisfies the following conditions:*

1. *for each $Y \in S_{n+1}$ there exists $X \in S_n$ such that $Y \in \vartheta(X)$;*
2. *if $X_1, X_2 \in S_n$ and $X_1 \neq X_2$, then $\vartheta(X_1) \bigcap \vartheta(X_2) = \emptyset$;*

then, the family of sets $\mathcal{F}_{n+1} = \{\vartheta(X) : \forall X \in S_n\}$ is a partition of S_{n+1}. The operator ϑ will be called ECO operator.

A *succession rule* Ω, as said above, can be useful to encode the ECO construction of a class of objects. Usually, it looks as a formal system as follows:

$$\Omega : \begin{cases} (a) \\ (k) \rightsquigarrow (b_1(k))(b_2(k))\dots(b_k)(k) \end{cases}$$

where (a), (k) and $(b_i(k))$ are the *labels* of the rule (positive integers). Each objects of the class has a label which generally represents the cardinality of the set of objects which can be generated from it. In a few words, the label of an objects is the number of its sons. The *axiom* of the rule is (a) and is the label of the minimal object of the class, with respect to the parameter we are basing on for the enumeration of the objects. The second line of the succession rule is said *production* of the label (k) (note that in a succession rule the productions can be more than one). A rule Ω is often figured with a *generating tree*, where (a) is the root and each node is a label (k) having k sons whose labels are those ones specified in the production of (k).

Chapter 2

Enumeration of generalized pattern avoiding permutations

The results of the first part of this chapter (Sections 2.2, 2.3, 2.4) concern the exact enumeration of the permutations, according to their length, avoiding any set of four or five generalized patterns [BS] of type $(1, 2)$ or $(2, 1)$. The cases of the permutations avoiding one, two or three generalized patterns (of the same types) were solved in [C], [CM] and [BFP], respectively. In particular, in [CM] the authors conjectured the plausible sequences enumerating the permutations of $S_n(P)$, for any set P of three or more patterns.

In [BFP], the proofs were substantially conducted by finding the ECO construction [BDPP1] for the permutations avoiding three generalized patterns of type $(1, 2)$ or $(2, 1)$, encoding it with a succession rule and, finally, checking that this one leads to the enumerating sequence conjectured in [CM]. This approach could be surely used also for the investigation of the avoidance of four or five generalized patterns of type $(1, 2)$ or $(2, 1)$ and, maybe, it would allow to find same nice and interesting results: we think that, for instance, in same case new succession rules for known sequences would appear. Nevertheless, this approach has just one obstacle: the large number of cases to consider in order to exhaust all the conjectures in [CM]. The line we are going to follow (see below) is simple and allows us to reduce the number of cases to be considered. Most of the results are summarized in several tables which are presented in the pages of the chapter. Really, this work could appear an easy exercise, but we believe that it is a valuable contribute to the classification of permutations avoiding generalized patterns, started with Claesson, Mansour, Elizalde and Noy [EN], Kitaev [K]. Moreover, it can be seen as the continuation of the work started in [BFP] for the fulfillment of the proofs of the conjectures presented in [CM].

2.1 The strategy

Looking at the table of [CM] where the authors present their conjectures, it is possible to note that most of the sequences enumerating the permutations avoiding four patterns are the same of those ones enumerating the permutations avoiding three patterns. A similar fact happens when the forbidden patterns are four and five. This suggests to use the results for the case of three forbidden patterns (at our disposal) to deduce the proof of the conjectures for the case of four forbidden patterns and, similarly, use the results for the case of four forbidden patterns to solve the case of five forbidden patterns. Indeed, it is obvious that $S(p_1, p_2, p_3, p_4) \subseteq S(p_{i_1}, p_{i_2}, p_{i_3})$ (with $i_j \in \{1, 2, 3, 4\}$ and $p_l \in \mathcal{M}$). If the inverse inclusion can be proved for some patterns, then the classes $S(p_1, p_2, p_3, p_4)$ and $S(p_{i_1}, p_{i_2}, p_{i_3})$ coincide and they are enumerated by the same sequence (a similar argument can be used for the case of four and five forbidden patterns).

The following eight propositions are useful to this aim, as well: each of them proves that if a permutation avoids certain patterns, than it avoids also a further pattern. Therefore, it is possible to apply one of them to a certain class $S(p_{i_1}, p_{i_2}, p_{i_3})$ to prove that $S(p_{i_1}, p_{i_2}, p_{i_3}) \subseteq S(p_1, p_2, p_3, p_4)$ (the generalization to the case of four and five forbidden pattern is straightforward). The proof of the first four of them can be found in [BFP].

Proposition 2.1.1 *If* $\pi \in S(2-13)$, *then* $\pi \in S(2-13, 21-3)$.

Proposition 2.1.2 *If* $\pi \in S(31-2)$, *then* $\pi \in S(31-2, 3-12)$.

Proposition 2.1.3 *If* $\pi \in S(2-31)$, *then* $\pi \in S(2-31, 23-1)$.

Proposition 2.1.4 *If* $\pi \in S(13-2)$, *then* $\pi \in S(13-2, 1-32)$.

Proposition 2.1.5 *If* $\pi \in S(1-23, 2-13)$, *then* $\pi \in S(1-23, 2-13, 12-3)$.

Proof. Suppose that π contains a $12-3$ pattern in the entries π_i, π_{i+1} and π_k $(k > i+1)$. Let us consider the entry π_{i+2}. It can be neither $\pi_{i+2} > \pi_{i+1}$ (since $\pi_i\ \pi_{i+1}\ \pi_{i+2}$ would show a pattern $1-23$) nor $\pi_{i+2} < \pi_{i+1}$ (since $\pi_{i+1}\ \pi_{i+2}\ \pi_k$ would show a pattern $21-3$ which is forbidden thanks to Proposition 2.1.1).

□

(The proof of the following proposition is very similar and is omitted.)

Proposition 2.1.6 *If* $\pi \in S(1-23, 21-3)$, *then* $\pi \in S(1-23, 21-3, 12-3)$.

Proposition 2.1.7 *If* $\pi \in S(1-23, 2-31)$, *then* $\pi \in S(1-23, 2-31, 12-3)$.

Proof. Suppose that a pattern $12-3$ appear in π_i, π_{i+1} and π_k. If we consider the entry π_{k-1}, then it is easily seen that it can be neither $\pi_i < \pi_{k-1} < \pi_k$ (the entries $\pi_i\ \pi_{k-1}\ \pi_k$ would be $1-23$ pattern like) nor $\pi_{k-1} < \pi_i$ (the entries $\pi_i\ \pi_{i+1}\ \pi_{k-1}$ would show a pattern $23-1$ which is forbidden thanks to Proposition 2.1.3). Hence, $\pi_{k-1} > \pi_k$. We can repeat the same above argument for the entry π_j, $j = k-2, k-3, \ldots, i+2$, concluding each time that $\pi_j > \pi_{j+1}$. When $j = i+2$ a pattern $1-23$ is shown in $\pi_i\ \pi_{i+1}\ \pi_{i+2}$, which is forbidden.

□

Proposition 2.1.8 *If* $\pi \in S(1{-}23, 23{-}1)$*, then* $\pi \in S(1{-}23, 23{-}1, 12{-}3)$.

This last proposition can be be proved by simply adapting the argument of the proof of the preceding one.

2.2 Permutations avoiding four patterns

First of all we recall the results of [BFP] in Tables 2.1 and 2.2. For the seek of brevity, for each symmetry class only a representative is reported. In the first column of these tables, a name to each symmetry class is given (as in [BFP]), the second one shows the three forbidden patterns (the representative) and the third one indicates the sequence enumerating the permutations avoiding the specified patterns.

Having at our disposal the results for the permutations avoiding three patterns, the proofs for the case of four forbidden patterns are conducted following the line indicated in the previous section. These proofs are all summarized in tables. Tables 2.3, 2.4 and 2.5 are related to the permutations avoiding four patterns enumerated by the sequences $\{n\}_{n\geq 1}$, $\{F_n\}_{n\geq 1}$ and $\{2^{n-1}\}_{n\geq 1}$, respectively (the succession F_n denotes the Fibonacci numbers). As in [BFP], the empty permutation with length $n = 0$ is not considered, therefore the length is $n \geq 1$. The tables have to be read as follows: consider the representative of the symmetry class specified in the rightmost column of each table; apply the proposition indicated in the precedent column to the three forbidden patterns which one can find in Tables 2.1 and 2.2 to obtain the four forbidden patterns written in the column named *avoided patterns*. At this point, as we explained in the previous section, the permutations avoiding these four patterns are enumerated by the same sequence enumerating the permutations avoiding the three patterns contained in the representative of the symmetry class indicated in the rightmost column.

The first column of Table 2.3 and 2.4 specifies a name for the the symmetry class represented by the four forbidden patterns of the second column. This name is useful in the next section. Table 2.12 indicates in the first column the sequence enumerating the permutations avoiding the patterns of the second column, which are obtained as in the above tables.

2.2.1 Classes enumerated by $\{0\}_{n\geq k}$.

The classes of four patterns avoiding permutations enumerated by the sequence $\{0\}_{n\geq k}$ can be handled in a very simple way. If $S(q_1, q_2, q_3)$, $q_i \in \mathcal{M}$, is a class of permutations avoiding three patterns such that $|S_n(q_1, q_2, q_3) = 0|$, for $n \geq k$, then it is easily seen that $S(q_1, q_2, q_3, r)$, $\forall r \in \mathcal{M}$, is also enumerated by the same sequence. Then, each symmetry class from $C1$ to $C7$ (see Table 2.2) generates nine symmetry classes by choosing the pattern $r \neq q_i$, $i = 1, 2, 3$. It is not difficult to see that all the classes we obtain in this way are not all different, thanks to the operations of reverse, complement and reverse-complement. In Table 2.6, only the different possible cases are presented. Here, the four forbidden patterns are recovered by adding a pattern of a box of the second column to the three patterns specified in the box to its right at the same level (rightmost column). The representative so obtained is recorded in the leftmost column with a name, which will be useful in the next section.

2.2.2 Classes enumerated by $\{2\}_{n\geq 2}$.

The enumerating sequences encountered till now (see Tables 2.3, 2.4, 2.5, 2.12, 2.6) are all involved in the enumeration of some class of permutations avoiding three patterns (Tables 2.1, 2.2). Therefore, applying the eight propositions of the previous section to the classes of Table 2.1 and 2.2, the three forbidden patterns have been increased by one pattern, obtaining Table 2.3, 2.4, 2.5, 2.12 and 2.6. For the classes enumerated by the sequence

$\{2\}_{n\geq 2}$ it is not possible to use the same strategy, since there are no classes of permutations avoiding three patterns enumerated by that sequence. The proofs, in this case, use four easy propositions whose proofs can be directly derived from the statement of the first four propositions of Section 2.1. We prefer to explicit them the same.

Proposition 2.2.1 *If a permutation π contains the pattern $23-1$, then it contains the pattern $2-31$, too.*

Taking the reverse, the complement and the reverse-complement of the patterns involved in Prop. 2.2.1, the following propositions are obtained:

Proposition 2.2.2 *If a permutation π contains the pattern $1-32$, then it contains the pattern $13-2$, too.*

Proposition 2.2.3 *If a permutation π contains the pattern $21-3$, then it contains the pattern $2-13$, too.*

Proposition 2.2.4 *If a permutation π contains the pattern $3-12$, then it contains the pattern $31-2$, too.*

In Table 2.7 the results relating to the enumeration of the permutations avoiding four patterns enumerated by the sequence $\{2\}_{n\geq 2}$ (whose proofs are contained in the six next propositions) are summarized. The four forbidden patterns can be recovered by choosing one pattern from each column, in the same box-row of the table.

In the sequel, $p_i \in A_i$ with $i = 1, 2, 3, 4$ where A_i is a subset of generalized patterns.

Proposition 2.2.5 *Let $A_1 = \{1-23\}$, $A_2 = \{2-31, 23-1\}$, $A_3 = \{1-32, 13-2\}$ and $A_4 = \{3-12, 31-2\}$. Then $|S_n(p_1, p_2, p_3, p_4)| = 2$ and $S_n = \{n\ (n-1)\dots 3\ 2\ 1,\ (n-1)\ (n-2)\dots 3\ 2\ 1\ n\}$.*

Proof. Let $\sigma \in S_n(p_2, p_3)$. Then, $\sigma_1 = n$ or $\sigma_n = n$, otherwise, if $\sigma_i = n$ with $i \neq 1, n$, the entries $\sigma_{i-1}\sigma_i\sigma_{i+1}$ would be a forbidden pattern p_2 or p_3.

If $\rho \in S_n(p_1, p_3)$, then $\rho_{n-1} = 1$ or $\rho_n = 1$, otherwise, if $\rho_i = 1$ with $i < n-1$, then the entries $\rho_i\rho_{i+1}\rho_{i+2}$, would be a forbidden pattern p_1 or p_3.

Therefore, if $\pi \in S_n(p_1, p_2, p_3)$, then there are only the following three cases for π:

1. $\pi_n = n$ and $\pi_{n-1} = 1$. In this case $\pi = (n-1)\ (n-2)\ldots 2\ 1\ n$, otherwise, if an ascent appears in $\pi_j\pi_{j+1}$ with $j = 1, 2, \ldots, n-3$, the entries $\pi_j\pi_{j+1}\pi_{n-1}$ would show the pattern $23-1$ and π would contain the pattern $2-31$, too (see Prop. 2.2.1).

2. $\pi_1 = n$ and $\pi_n = 1$. In this case $\pi = n\ (n-1)\ldots 3\ 2\ 1$, otherwise, if an ascent appears in $\pi_j\pi_{j+1}$ with $j = 2, 3, \ldots, n-2$, the entries $\pi_j\pi_{j+1}\pi_n$ would show the pattern $23-1$ and π would contain the pattern $2-31$, too (see Prop. 2.2.1).

3. $\pi_1 = n$ and $\pi_{n-1} = 1$ (and $\pi_n = k < n$).

If π has to avoid the pattern p_4, too ($\pi \in S_n(p_1, p_2, p_3, p_4)$), then the third above case is not allowed since $\pi_1\pi_{n-1}\pi_n$ are a $3-12$ pattern which induces an occurrence of $31-2$ in π (Prop. 2.2.4).

□

Proposition 2.2.6 *Let* $A_1 = \{1-23\}$, $A_2 = \{2-13, 21-3\}$, $A_3 = \{1-32, 13-2\}$ *and* $A_4 = \{3-12, 31-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n-1)\ldots 3\ 2\ 1,\ (n-1)\ n\ (n-2)\ (n-3)\ldots 2\ 1\}$.

Proof. If $\sigma \in S_n(p_1, p_2)$, then $\pi_1 = n$ or $\pi_2 = n$. If $\rho \in S_n(p_1, p_3)$, then $\pi_n = 1$ or $\pi_{n-1} = 1$. Then, if $\pi \in S_n(p_1, p_2, p_3)$, there are only the four following cases:

1. $\pi_1 = n$ and $\pi_n = 1$.

2. $\pi_2 = n$ and $\pi_n = 1$. In this case $\pi_1 = n - 1$, otherwise if $\pi_k = n - 1$ with $k > 3$, then $\pi_{k-2}\pi_{k-1}\pi_k$ is a $1 - 23$ pattern or a $21 - 3$ pattern which induces an occurrence of $2 - 13$ (Prop. 2.2.3). If $k = 3$, then $\pi_1\pi_2\pi_3$ is a $1 - 32$ or $13 - 2$ pattern which are forbidden.

3. $\pi_1 = n$ and $\pi_{n-1} = 1$.

4. $\pi_2 = n$ and $\pi_{n-1} = 1$. For the same reasons of case 2, it is $\pi_1 = n - 1$.

If π has to avoid p_4, too ($\pi \in S_n(p_1, p_2, p_3, p_4)$), then the third and the fourth above cases are not allowed since $\pi_1\pi_{n-1}\pi_n$ are a $3-12$ pattern which induces an occurrence of $31 - 2$ (Prop. 2.2.4). Moreover, the permutations of the above cases 1 and 2, must be such that there are not ascents $\pi_i\pi_{i+1}$ between n and 1 in order to avoid p_4. Then, $\pi = n\ (n - 1) \ldots 3\ 2\ 1$ or $\pi = (n - 1)\ n\ (n - 2) \ldots 3\ 2\ 1$.

□

Proposition 2.2.7 *Let* $A_1 = \{2 - 13, 21 - 3\}$, $A_2 = \{2 - 31, 23 - 1\}$, $A_3 = \{1-32, 13-2\}$ *and* $A_4 = \{3-12, 31-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n - 1) \ldots 2\ 1,\ 1\ 2 \ldots n\}$.

Proof. It is easily seen that each three consecutive elements of π can only be in increasing or decreasing order.

□

Proposition 2.2.8 *Let* $A_1 = \{12 - 3\}$, $A_2 = \{2 - 13, 21 - 3\}$, $A_3 = \{2 - 31, 23 - 1\}$ *and* $A_4 = \{32 - 1\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{1\ n\ 2\ (n - 1) \ldots,\ n\ 1\ (n - 1)\ 2 \ldots\}$.

Proof. If $\pi \in S_n(p_1, p_2, p_3, p_4)$, then it is easy to see that $\pi_1\pi_2 = 1\ n$ or $\pi_1\pi_2 = n\ 1$. Considering the sub-permutation $\pi_2\pi_3 \ldots \pi_n$, in the same way we deduce $\pi_2\pi_3 = 2\ (n-1)$ or $\pi_2\pi_3 = (n-1)\ 2$. The thesis follows by recursively using the above argument.

□

Proposition 2.2.9 *Let* $A_1 = \{1-23\}$, $A_2 = \{2-13, 21-3\}$, $A_3 = \{2-31, 23-1\}$ *and* $A_4 = \{3-12, 31-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n-1)\dots 1,\ 1\ n\ (n-1)\dots 3\ 2\}$.

Proof. Let $\pi \in S_n(p_1, p_2, p_3, p_4)$. It is $\pi_1 = n$ or $\pi_2 = n$, otherwise a $1-23$ or p_2 pattern would appear.

If $\pi_1 = n$, then $\pi = n\ (n-1)\dots 1$ since if an ascent appears in $\pi_i\pi_{i+1}$, the entries $\pi_1\pi_i\pi_{i+1}$ are a p_4 pattern.

If $\pi_2 = n$, then $\pi_1 = 1$ since the p_3 pattern has to be avoided. Moreover, in this case, it is $\pi_j > \pi_{j+1}$ with $j = 3, 4, \dots, (n-1)$ in order to avoid $1-23$. Then $\pi = 1\ n\ (n-1)\dots 2\ 1$.

□

Proposition 2.2.10 *Let* $A_1 = \{1-23\}$, $A_2 = \{2-13, 21-3\}$, $A_3 = \{2-31, 23-1\}$ *and* $A_4 = \{1-32, 13-2\}$. *Then* $|S_n(p_1, p_2, p_3, p_4)| = 2$ *and* $S_n = \{n\ (n-1)\dots 3\ 2\ 1,\ n\ (n-1)\dots 3\ 1\ 2\}$.

Proof. Let $\pi \in S_n(p_1, p_2, p_3, p_4)$. The entries 1 and 2 have to be adjacent in order to avoid p_3 and p_4 and $\pi_n = 1$ or $\pi_{n-1} = 1$ in order to avoid p_1 and p_4. So, $\pi_{n-1}\pi_n = 1\ 2$ or $\pi_{n-1}\pi_n = 2\ 1$. Moreover, each couple of adjacent elements $\pi_j\pi_{j+1}$ must be a descent, otherwise a $23-1$ pattern (which induces an occurrences of $2-31$) would appear. Then $\pi = n\ (n-1)\dots 3\ 2\ 1$ or $\pi = n\ (n-1)\dots 3\ 1\ 2$.

□

The conjecture stated in [CM] about the permutations enumerated by $\{2\}_{n\geq 2}$ declares that there are 42 symmetry classes of such permutations, while from Table 2.7 it is possible to deduce 52 symmetry classes. Nevertheless, it is not difficult to check that these classes are not all different: for example the symmetry class $\{2-13, 2-31, 1-32, 31-2\}$ is the same of $\{2-13, 23-1, 13-2, 31-2\}$ (the second one is the reverse of the first one). Note that the repetitions come out only from the third box-row of Table 2.7.

2.3 Permutations avoiding five patterns

2.3.1 Classes enumerated by $\{1\}_{n\geq 1}$

The sequence $\{1\}_{n\geq 1}$ does not enumerate any class of permutations avoiding four patterns, so that we can not apply the same method of the previous section using the proposition of the Introduction.

Referring to Proposition 2.2.7, we deduce that there are sixteen different classes $S_n(p_1, p_2, p_3, p_4)$ such that $p_i \in A_i$ with $i = 1, 2, 3, 4$. We recall that $|S_n(p_1, p_2, p_3, p_4)| = 2$ and $S_n(p_1, p_2, p_3, p_4) = \{n\ (n-1)\dots 2\ 1,\ 1\ 2\dots n\}$. If a permutation $\pi \in S_n(p_1, p_2, p_3, p_4)$ has to avoid the pattern $1-23$, too, then $\pi = n\ (n-1)\dots 2\ 1$ and $|S_n(p_1, p_2, p_3, p_4, 1-23)| = 1$.

Then, it is easy to see that the five forbidden patterns avoided by the permutations enumerated by $\{1\}_{n\geq 1}$ can be recovered by considering the four patterns chosen from the third box-row of Table 2.7 (one pattern from each column) and the pattern $1-23$. We do not present the relative table.

2.3.2 Classes enumerated by $\{0\}_{n\geq k}$

This case is treated as the case of the permutations avoiding four patterns. It is sufficient to add a pattern $r \in \mathcal{M}$ to each representative (from O1 to O37 in Table 2.6) of four forbidden patterns of Table 2.6 in order to obtain a representative T of five forbidden patterns such that $|S_n(T)| = 0, n \geq 4$. In Table 2.8 we present the different representatives T which can be derived from Table 2.6. The five forbidden patterns of each representative are a pattern chosen in a box of the first column and the four patterns indicated by the representative (which refer to Table 2.6) in the second box at the same level. In the table, only the different representatives of five patterns are presented.

2.3.3 Classes enumerated by $\{2\}_{n\geq k}$, $\{n\}_{n\geq 1}$, $\{F_n\}_{n\geq 1}$

Tables 2.9 and 2.10 summarize the results related to the permutations avoiding five patterns enumerated by $\{2\}_{n\geq k}$. The five forbidden patterns are obtained by considering a representative of four forbidden patterns of the rightmost column and the pattern specified in the corresponding box of the preceding column. The first column indicates which is the proposition to apply. Note that each representative of four patterns (rightmost column) can be found in Table 2.7.

The reading of Tables 2.11 and 2.13 (related to the sequences $\{n\}_{n\geq 1}$ and $\{F_n\}_{n\geq 1}$, respectively) is as usual: apply the proposition specified in the first column to recover the representative of five forbidden patterns which is composed by the pattern of the second column and the four patterns of the representative indicated in the rightmost column. Here, the names of the representatives refer to Tables 2.3 and 2.4.

2.4 Conclusion: the cases of more than five patterns

The approach we have followed in this work can be used to investigate the enumeration of the permutations avoiding more than five patterns. Really, applying the same propositions (we have herein used) to the results about the case of the avoidance of five patterns, one can try to solve the conjectures for the case of six patterns. The successive cases can be examined in a similar way.

The case of six patterns is the unique, among the remaining, which presents some enumerating sequence not definitively constant (as it can be checked by looking at the tables of the conjectures in [CM]). We note also that all these sequences appear in the enumeration of the case of five patterns. If $|S_n(P)|$ is required, with $P \subseteq \mathcal{M}$, $|P| = 6$, it should take a few minutes to find the set Q of five generalized patterns such that the application of a certain proposition on Q (among those ones presented in this work) leads to the set P of six forbidden patterns. So $|S_n(Q)| = |S_n(P)|$. Clearly, we are not sure that such a set Q exists since the statements in [CM] are only conjectures. Moreover, it is not sure even the fact that any subset P could be obtained by applying some proposition to some patterns of $Q \subset P$. Nevertheless, the application of the above mentioned propositions to the sets Q of five forbidden patterns should be confirm most of the conjectures about the case of six patterns. This is the reason why we did not present the analysis of this case, together with the fact that several other tables would have appeared in these pages!

To conclude, we think that a further work about the cases of more than six forbidden patterns does not seem to be necessary, since many of the remaining conjectures in [CM] can be easily proved. Moreover, if $S_n(P)$ is needed, with $|P| > 6$, an argument similar to the case $|P| = 6$ can be done.

2.5 Some statistics on permutations avoiding generalized patterns

2.5.1 Preliminaries

In the last decade a huge amount of articles has been published studying pattern avoidance on permutations. From the point of view of enumeration, typically one tries to count permutations avoiding certain patterns according to their lengths. Here we tackle the problem of refining this enumeration by considering the statistics "first/last entry". We give complete results for every generalized patterns of type $(1,2)$ or $(2,1)$ as well as for some cases of permutations avoiding a pair of generalized patterns of the above types.

The twelve generalized patterns of $\mathcal{P}$ are organized in three symmetry classes : $\{1-23, 32-1, 3-21, 12-3\}$, $\{3-12, 21-3, 1-32, 23-1\}$ and $\{2-13, 31-2, 2-31, 13-2\}$. If p and p' are two patterns such that $|S_n(p)| = |S_n(p')|$, then p and p' are said to be in the same *Wilf class* [M1]. Since in [C] it is shown that

- $|S_n(p)| = B_n$,
  for $p \in \{1-23, 32-1, 3-21, 12-3\} \bigcup \{3-12, 21-3, 1-32, 23-1\}$
- $|S_n(p)| = C_n$,
  for $p \in \{2-13, 31-2, 2-31, 13-2\}$,

where B_n and C_n are the n-th Bell and Catalan numbers, respectively, then we can say that $\mathcal{P}$ is organized in two Wilf classes: $\{1-23, 32-1, 3-21, 12-3, 3-12, 21-3, 1-32, 23-1\}$ and $\{2-13, 31-2, 2-31, 13-2\}$.

In this work, we refine some enumerative results on $S(p)$, $p \in \mathcal{P}$, namely we count p-avoiding permutations, for each p, according to their length and the value of their first or last entry. Next we solve the same problem for some classes of permutations of the kind $S(p,q)$, $p, q \in \mathcal{P}$, and we conclude by proposing to tackle this problem for any remaining pair of generalized patterns of $\mathcal{P}$.

Our results are achieved by using the ECO method together with a graphical representation of permutations. In the following we only briefly recall the ECO construction for (patterns avoiding) permutations, for more details we refer the reader to [BDPP1] and [BFP].

Any permutation of length n can be visualized using a path-like representation, as in Figure 2.1. Note that the plane is divided in $n+1$ strips

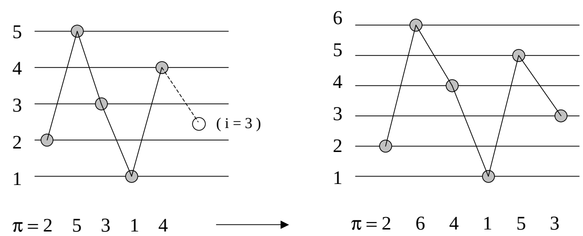


Figure 2.1 An ECO construction for permutations

by the n horizontal lines which are numerated from 1 to n, starting from bottom (in the sequel, we refer to these strips as "regions": region i is included between line $i-1$ and line i, whereas region 1 is the one below line 1 and region $n+1$ is the one above line n). Each entry of the permutation is represented as a "node" lying on the line corresponding to its value. If $\pi \in S_n$, then $n+1$ permutations belonging to S_{n+1} can be obtained by inserting a new node in each region of the plane. If we wish to generate the permutations in $S_{n+1}(P)$ obtained in such a way from $\pi \in S_n(P)$, where P is a set of forbidden patterns, then the regions the last node can be inserted in form a subset of all the $n+1$ possible regions; in the framework of the ECO method they are called *active sites* [BDPP1]. A remarkable feature of this construction is that, if $\pi \in S_n(P)$, then $\pi' \in S_{n+1}$ (which is obtained from π by inserting the last node in one of the regions) does not contain the patterns specified in P in its entries π'_j with $j = 1, \ldots, n$, otherwise π itself would contain some pattern of P. So, to decide if a region i is an active site or not, we just have to check those generalized patterns the last node is involved in.

2.5.2 The symmetry class $\{1-23, 32-1, 3-21, 12-3\}$

ECO construction and generating tree of $S(1-23)$

Let $\pi \in S_n(1-23)$. If $\pi_n = k \neq 1$, then π generates k permutations $\pi^{(i)} \in S_{n+1}(1-23)$, $i = 1, 2, \ldots, k$, by inserting a new node in region i. If $\pi_n = 1$, then π generates $n+1$ permutations by inserting a new node in any region. Note that in this case the number of sons of π is determined by the length of π. If $\pi^{(r)} \in S_{n+1}(1-23)$ denotes the permutation of $S_{n+1}(1-23)$ derived from $\pi \in S_n(1-23)$ by inserting the last node in region r, it is easily seen that $\pi^{(1)}$ generates, in turns, $n+2$ permutations, whereas $\pi^{(r)}$, $r \neq 1$, produces r permutations of $S_{n+2}(1-23)$. This ECO construction can be represented as in Figure 2.2 and, if we label with (k, n) each permutation of $S_n(1-23)$ having k active sites, it can be encoded by the following succession rule:

$$\Omega : \begin{cases} (2,1) \\ (k,n) \rightsquigarrow (2,n+1)(3,n+1)\cdots(k,n+1)(n+2,n+1) \end{cases} .$$

We now wish to draw the generating tree related the the previous succession rule. For the sake of simplicity and for reasons that will become clear later, we choose to label the nodes of the generating tree using the number of their sons, which correspond to the first element of each label of the succession rule. In Figure 2.3 we have depicted the first levels of the generating tree of $S(1-23)$. Here the labels in bold character correspond to the labels of the kind $(n+1, n)$ in the succession rule. Observe that the production of each label depends on its level in the generating tree.

Distribution according to the length and the last value

Starting from the generating tree of Figure 2.3, we can consider the matrix $M = (m_{ij})_{i,j\geq 1}$ where $m_{i,j}$ is the number of labels $j+1$ at level i in the generating tree:

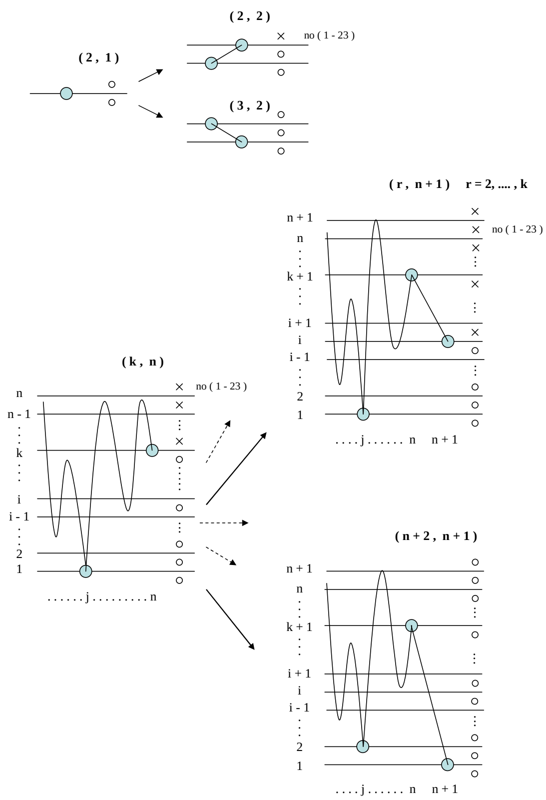


Figure 2.2 ECO construction of $S(1-23)$

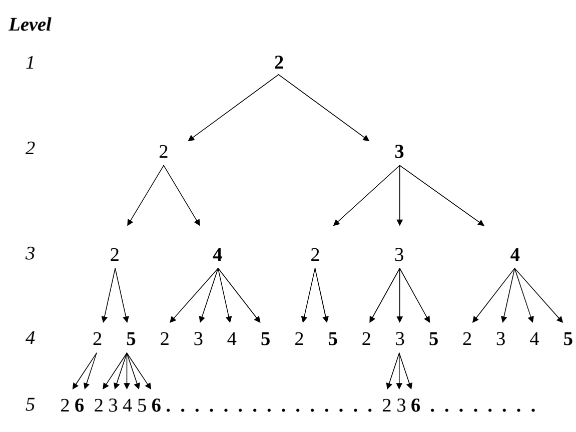


Figure 2.3 The generating tree of $S(1-23)$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 1 & 2 & 0 & 0 & 0 & \vdots \\ 5 & 3 & 2 & 5 & 0 & 0 & \vdots \\ 15 & 10 & 7 & 5 & 15 & 0 & \vdots \\ 52 & 37 & 27 & 20 & 15 & 52 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

The above matrix M is called the *ECO matrix* of the rule Ω, according to [DFR1]. It is easily seen that M can be recursively described as follows:

1. $m_{1,1} = 1$ (the minimal permutation $\pi = 1$ has two sons);
2. $m_{n,k} = 0$ if $k > n$ (each permutation of length n has at most n sons);
3. $m_{n,k} = \sum_{i=k}^{n-1} m_{n-1,i}$ if $k < n$ (this derives directly from the recursive interpretation of the previous succession rule);
4. $m_{n,n} = m_{n,1}$ (each permutation of length $n-1$ produces precisely one son having label 2 and precisely one son having label $n+1$).

Since $m_{n,1} (= m_{n,n})$ is the sum of all the elements in the $(n-1)$-th row (for $n > 1$), this entry records the total number of $(1-23)$-avoiding permutations of length $n-1$. In other words, $m_{n,1} = B_{n-1}$.

Moreover, from a careful inspection of M, we have that $m_{n,k-1}$, with $k = 2, \dots, n$, is the number of permutations of $S_n(1-23)$ ending with k and $m_{n,n}$ is the number of permutations of $S_n(1-23)$ ending with 1. Then, if we move the diagonal of M such that it becomes the first column of the matrix, we obtain a new matrix $A = (a_{i,j})_{i,j\geq 1}$ where $a_{i,j}$ is the number of $(1-23)$-avoiding permutations of length i ending with j.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 5 & 5 & 3 & 2 & 0 & 0 & \vdots \\ 15 & 15 & 10 & 7 & 5 & 0 & \vdots \\ 52 & 52 & 37 & 27 & 20 & 15 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

The matrix A is essentially the *Bell triangle*, which can be found in [W] together with several other references.

The above recursive properties of M can be immediately translated as follows:

1. $a_{1,1} = 1$ (the minimal permutation ends, trivially, with 1);
2. $a_{n,k} = 0$ if $k > n$ (each permutation of length n cannot end with a number greater than n itself);
3. $a_{n,k} = \sum_{i=k}^{n-1} a_{n-1,i} + a_{n-1,1}$ if $2 \leq k \leq n$ (the diagonal of M has been moved in the first column of A);
4. $a_{n,1} = a_{n,2}$ (since $a_{n,1} = m_{n,n} = m_{n,1} = a_{n,2}$).

From 3 we obtain, for $k \geq 3$:

$$a_{n,k} = a_{n,k-1} - a_{n-1,k-1},$$

If we denote by ∇ the usual *backward difference operator*, since $a_{n,2} = B_{n-1}$, we get:

$$\begin{aligned} a_{n,k} &= \nabla a_{n,k-1} \\ &= \nabla^2 a_{n,k-1} \\ &= \cdots \\ &= \nabla^{k-2} a_{n,2} = \nabla^{k-2} B_{n-1} \qquad \text{(which holds also for } k = 2\text{)}. \end{aligned}$$

Thus we find the following formulas concerning the distribution of $1-23$-avoiding permutations according to their length and to the value of their last entry:

$$|\{\pi \in S_n(1-23) : \pi_n = 1\}| = B_{n-1},\ n \geq 1;$$

$$|\{\pi \in S_n(1-23) : \pi_n = k\}| = \nabla^{k-2}(B_{n-1}),\ 2 \leq k \leq n.$$

The other patterns of the class

The arguments employed for $S(1-23)$ can be easily modified for the other patterns of the symmetry class of $1-23$, obtaining similar results. The ECO construction, in these cases, has to be adapted in order to obtain the same succession rule and the same generating tree we got for $S(1-23)$. The matrices M and A are defined as in the previous section.

1. For the reverse pattern of $1-23$, i.e. $32-1$, we find that $a_{i,j}$ is the number of permutations π of length i such that $\pi_1 = j$, and so:

   - $|\{\pi \in S_n(32-1) : \pi_1 = 1\}| = B_{n-1},\ n \geq 2$;
   - $|\{\pi \in S_n(32-1) : \pi_1 = k\}| = \nabla^{k-2}(B_{n-1}),\ 2 \leq k \leq n.$

   Note that in this case the ECO construction can be, in some way, "reversed", so that the active sites are not on the right of the diagram of the permutation π but on its left, i.e. before the first entry of π.

2. For the complement pattern $3-21$, we have that $a_{i,j}$ is the number of permutations of length i ending with $i+1-j$:

   - $|\{\pi \in S_n(3-21) : \pi_n = n\}| = B_{n-1}, n \geq 1;$
   - $|\{\pi \in S_n(3-21) : \pi_n = k\}| = \nabla^{n-k-1}(B_{n-1}),\ 1 \leq k \leq n-1.$

3. For the reverse-complement pattern $12-3$, $a_{i,j}$ is the number of permutations π of length i such that $\pi_1 = i+1-j$, and so:

   - $|\{\pi \in S(12-3) : \pi_1 = n\}| = B_{n-1},\ n \geq 1;$

- $|\{\pi \in S(12-3) : \pi_1 = k\}| = \nabla^{n-k-1}(B_{n-1}),\ \ 1 \leq k \leq n-1.$

2.5.3 The symmetry class $\{3-12, 21-3, 1-32, 23-1\}$

ECO construction and generating tree of $S(3-12)$

Let $\pi \in S_n(3-12)$. If $\pi_n = k-1 \neq n$, then π generates k permutations $\pi^{(i)} \in S_{n+1}(3-12)$, $i = 1, 2, \ldots, k-1, n+1$, by inserting a new node in region i. If $\pi_n = n$, then π generates $n+1$ permutations by inserting a new node in any region. As it happened for the class $S(1-23)$, note that the number of sons of π is determined by the length of π. It is easily seen that $\pi^{(n+1)}$ generates, in turns, $n+2$ permutations, whereas $\pi^{(i)}$ $(i \neq n+1)$ produces $i+1$ permutations. This ECO construction is illustrated in Figure 2.4. If each permutation of $S_n(3-12)$ with k active sites is labelled (k, n), then such a construction can be encoded using the following succession rule:

$$\begin{cases} (2,1) \\ (k,n) \rightsquigarrow (2,n+1)(3,n+1)\cdots(k,n+1)(n+2,n+1) \end{cases} .$$

Since it is the same succession rule we got for $S(1-23)$, the generating tree for $S(3-12)$ can be obtained in the same way.

Distribution according to the length and the last value

Defining the matrix $M = (m_{ij})_{i,j\geq 1}$ as in Section 2.5.2, it can be easily deduced that $m_{n,k}$ is the number of permutations of $S_n(3-12)$ ending with k. Note that in this case we do not need to move the diagonal of M to obtain the final matrix. Therefore, using again the backward difference operator ∇, the entries of M have the form:

$$m_{n,k} = \nabla^{k-1}(B_{n-1})$$

whence:

$$|\{\pi \in S_n(3-12) : \pi_n = n\}| = B_{n-1},\ \ n \geq 2\ ;$$

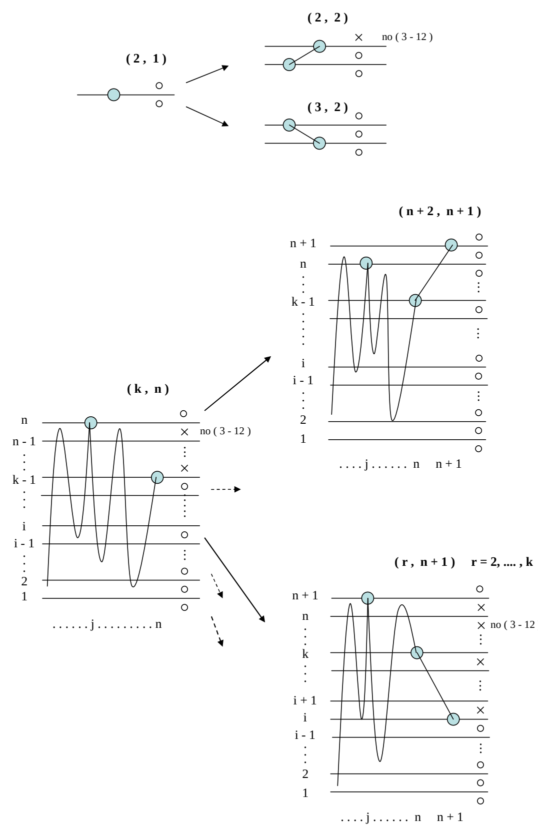


Figure 2.4 ECO construction of $S(3-12)$

$$|\{\pi \in S_n(3-12) : \pi_n = k\}| = \nabla^{k-1}(B_{n-1}),\ 1 \le k \le n-1\ .$$

The other patterns of the class

Proceeding as in Section 2.5.2, we get:

- $|\{\pi \in S_n(21-3) : \pi_1 = n\}| = B_{n-1},\ n \ge 2;$
- $|\{\pi \in S_n(21-3) : \pi_1 = k\}| = \nabla^{k-1}(B_{n-1}),\ 1 \le k \le n-1;$
- $|\{\pi \in S_n(1-32) : \pi_n = 1\}| = B_{n-1},\ n \ge 2;$
- $|\{\pi \in S_n(1-32) : \pi_n = k\}| = \nabla^{n-k}(B_{n-1}),\ 2 \le k \le n\ ;$
- $|\{\pi \in S_n(23-1) : \pi_1 = 1\}| = B_{n-1},\ n \ge 2\ ;$
- $|\{\pi \in S_n(23-1) : \pi_1 = k\}| = \nabla^{n-k}(B_{n-1}),\ 2 \le k \le n.$

2.5.4 The symmetry class $\{2-13, 31-2, 2-31, 13-2\}$

The permutations of $S(2-13)$ are enumerated by Catalan numbers [C]. As far as the ECO construction of $S(2-13)$ is concerned, we just note that, if $\pi \in S_n(2-13)$ is such that $\pi_n = k$, then region i, for $i = 1, 2, \dots, k+1$, is an active site for π. The succession rule encoding this construction is:

$$\begin{cases} (2) \\ (k) \rightsquigarrow (2)(3)\cdots(k+1) \end{cases}$$

Defining the matrix M as in the preceding sections, we obtain

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 5 & 5 & 3 & 1 & 0 & 0 & \vdots \\ 14 & 14 & 9 & 4 & 1 & 0 & \vdots \\ 42 & 42 & 28 & 14 & 5 & 1 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

which is the well-known Catalan Triangle whose entries $m_{i,j} = \frac{j}{i}\binom{2i-j-1}{i-1}$ are the ballot numbers and whose properties can be found, for example, in [NZ].

In the following, we present the results for all the patterns of the class, which can be derived as in the previous sections (the $m_{n,k}$'s are defined as before):

- $|\{\pi \in S_n(2-13) : \pi_n = k\}| = m_{n,k} = \frac{k}{n}\binom{2n-k-1}{n-1}$;
- $|\{\pi \in S_n(31-2) : \pi_1 = k\}| = m_{n,k} = \frac{k}{n}\binom{2n-k-1}{n-1}$;
- $|\{\pi \in S_n(2-31) : \pi_n = k\}| = m_{n,n-k+1} = \frac{n-k+1}{n}\binom{n+k-2}{n-1}$;
- $|\{\pi \in S_n(13-2) : \pi_1 = k\}| = m_{n,n-k+1} = \frac{n-k+1}{n}\binom{n+k-2}{n-1}$.

2.5.5 Permutations avoiding a pair of generalized patterns of type $(1,2)$ or $(2,1)$

In [CM] Claesson and Mansour counted permutations avoiding a pair of generalized patterns of type (1,2) or (2,1). Similarly to what we have done in the previous sections, we can study the distribution of the statistic "first/last entry" on permutations avoiding two or more generalized patterns. Here, we consider only two special examples, the former being quite easy, whereas the latter is surely more interesting. All the remaining cases are left to the readers as open problems for future research.

An easy case

We first deal with the permutations of $S(1-23, 1-32)$. This class is enumerated by the number I_n of involutions in S_n (see [CM]). An ECO construction of this class can be encoded by the following succession rule :

$$\Omega : \begin{cases} (2,1) \\ (1,n) \rightsquigarrow (n+2,n+1) \\ (n+1,n) \rightsquigarrow (1,n+1)^n(n+2,n+1) \end{cases}$$

where the first element in the label is the number of active sites of the permutation and the second one is its length. This can be checked by representing permutations by means of the usual path-like representation: indeed, if a permutation ends with 1, then an element can be inserted on its right in any region, whereas if a permutation ends with $k \neq 1$, then the only element which can be inserted must be placed in region 1 on the right. The reader is invited to complete the details, so to obtain the construction described precisely by the succession rule Ω.

From the generating tree of Ω, the matrix M whose entry $m_{i,j}$ is the number of vertices with label j at level i $(i, j \geq 1)$ can be constructed as in the preceding cases:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \vdots \\ 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & \vdots \\ 16 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & \vdots \\ 50 & 0 & 0 & 0 & 0 & 0 & 26 & 0 & \vdots \\ 156 & 0 & 0 & 0 & 0 & 0 & 0 & 76 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}.$$

The entries can be immediately computed as follows:

- $m_{1,1} = 0\ , \quad m_{1,2} = 1\ ;$
- $m_{n,1} = (n-1)m_{n-1,n}\ , \quad n \geq 2\ ;$
- $m_{n,n+1} = m_{n-1,1} + m_{n-1,n}\ , \quad n \geq 2\ ;$
- $m_{i,j} = 0$ in all the other cases.

From the ECO construction it easily appears that the first column of M counts the permutations π of $S_n(1{-}23, 1{-}32)$ such that $\pi_{n-1} = 1$ (or, which is the same, $\pi_n \neq 1$), whereas the super-diagonal sequence $m_{n,n+1}$ $(n \geq 1)$

Figure 2.5 The generating tree of $S(1-23, 21-3)$

shows the number of π ending with 1. Since if $\pi \in S_n(1-23, 1-32)$, then $\pi_{n-1} = 1$ or $\pi_n = 1$, we deduce that the super-diagonal satisfies $m_{n,n+1} = I_{n-1}$ $(n \geq 1)$.

A not so easy case

Our second example concerns the permutations of the class $S(1-23, 21-3)$, which also coincide with those of $S(1-23, 21-3, 12-3)$(see [BFP]) and are enumerated by Motzkin numbers. We will find the distribution of these permutations according to their length and their last entry; moreover, we will be able to derive the generating function of the sequences enumerating the permutations of this class whose last entry is k, for $k = 1, 2, \ldots$. We start by recalling the coloured succession rule Φ encoding an ECO construction for the above set of permutations (which can be found in [BFP]):

$$\Phi : \begin{cases} (\bar{2}) \\ (\bar{k}) \rightsquigarrow (\bar{2})(2)(3)\cdots(k) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(\overline{k+1}) \end{cases} .$$

In Figure 2.5, the first levels of the corresponding generating tree are presented.

As in the preceding examples, we construct a matrix $A = (a_{i,j})_{i,j\geq 1}$ recording in its entries the number of labels at each level of the tree: namely,

$a_{i,1}$ is the number of coloured label $\bar{k}$, $k \geq 2$, at level i of the tree and $a_{i,j}$, $j \geq 2$, is the number of labels j at level i. The first lines of A are:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 0 & 0 & 0 & 0 & \vdots \\ 4 & 4 & 1 & 0 & 0 & 0 & \vdots \\ 9 & 9 & 3 & 0 & 0 & 0 & \vdots \\ 21 & 21 & 8 & 1 & 0 & 0 & \vdots \\ 51 & 51 & 21 & 4 & 0 & 0 & \vdots \\ 127 & 127 & 55 & 13 & 1 & 0 & \vdots \\ 323 & 323 & 145 & 39 & 5 & 0 & \vdots \\ 835 & 835 & 385 & 113 & 19 & 1 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} .$$

As usual, we can find a recursive description of the entries of A:

- each label at level $i-1$ produces, among its sons, precisely one coloured label at level i, and so:

$$a_{i,1} = \sum_{r \geq 1} a_{i-1,r} \quad ;$$

- each label $j \geq 2$ at level i is generated either by a label $k \geq j$ at level $i-1$ or by a coloured label $\bar{k}$, with $k \geq j$ at level $i-1$, which, in turn, is generated by the label $k-1$ at level $i-2$, then:

$$a_{i,j} = \sum_{k \geq j} a_{i-1,k} + \sum_{k \geq j-1} a_{i-2,k} \quad \text{for} \quad j \geq 2 \; ; \qquad (2.1)$$

- it is easily seen that, in the above generating tree, the coloured label $\bar{k}$ first appears at the odd level $2k-3$, whereas the label k first appears at the even level $2k-2$, whence:

$$a_{i,j} = 0 \quad \text{for} \quad j \geq \lfloor i/2 \rfloor + 2 \; .$$

The ECO construction of $S(1-23,21-3)$ shows that, if a permutation has label k, then it ends with k, while if it has a coloured label $\bar{k}$, then its last entry is 1. Therefore, the entry $a_{i,j}$ is the number of permutations with length i and ending with the element j.

Our next aim is to find the generating function for the sequences displayed in the columns of the matrix A, which are the sequences enumerating the permutations of $S(1-23,21-3)$ with last entry $j=1,2,\ldots$, according to their length. It is convenient to change a little bit the notation: from now on, we will index the lines of A starting from 0 instead of 1. First of all, we derive a simple recurrence for the entries of A: using (2.1), simple calculations show that

$$a_{n,k}=a_{n,k-1}-a_{n-1,k-1}-a_{n-2,k-2}\ , \quad \text{for} \quad k\geq 2\ ,\ n\geq 0\ . \tag{2.2}$$

Let $C_k(x)$ be the generating function of the k-th column of A:

$$C_k(x)=\sum_{n\geq 0}a_{n,k}x^n\ .$$

Using (2.2), we find the following recurrence relation for $C_k(x)$:

$$C_{k+2}(x)=(1-x)C_{k+1}(x)-x^2C_k(x), \quad k\geq 0\ . \tag{2.3}$$

From the succession rule Φ (or from the ECO construction for $S(1-23,21-3)$), it is easy to check that

$$C_0(x)=M(x), \quad C_1(x)=M(x)-1\ ,$$

where

$$M(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

is the generating function of Motzkin numbers $\{M_n\}_{n\geq 0}$. In order to find a closed form for $C_k(x)$, we define a linear operator L on the vector space $\mathcal{V}$

of formal power series of odd order. The set $(C_k(x))_{k\geq 1}$ is a basis of $\mathcal{V}$, so L can be defined as follows:

$$L(C_k(x)) = C_{k+1}(x) \quad \text{for} \quad k \geq 1. \tag{2.4}$$

From (2.3) it is:

$$L^2(C_k(x)) = (1-x)L(C_k(x)) - x^2 C_k(x)$$

which is the same of

$$(L^2 - (1-x)L + x^2)C_k(x) = 0 \ .$$

Therefore the operator $L^2 - (1-x)L + x^2$ must vanish on $\mathcal{V}$. Solving the equation $L^2 - (1-x)L + x^2 = 0$, leads to

$$L = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2} = x^2 M(x) \ .$$

Now, from (2.4), we obtain the desired closed form for $C_k(x)$:

$$C_k(x) = x^2 M(x) C_{k-1}(x) = \cdots = x^{2(k-1)} M^{k-1}(x)(M(x) - 1), \ k \geq 1 \ .$$

symmetry class	avoided patterns	enumerating sequence
N1	{1-23,2-13,3-12}	
N2	{1-23,2-13,31-2}	
N3	{1-23,21-3,3-12}	
N4	{1-23,21-3,31-2}	
N5	{12-3,3-12,2-13}	
N6	{12-3,3-12,21-3}	
N7	{12-3,31-2,2-13}	
N8	{12-3,31-2,21-3}	
N9	{1-23,2-13,2-31}	
N10	{1-23,2-13,23-1}	
N11	{1-23,21-3,2-31}	
N12	{1-23,21-3,23-1}	
N13	{2-13,2-31,1-32}	$\{n\}_{n\geq 1}$
N14	{2-13,23-1,1-32}	
N15	{2-13,2-31,13-2}	
N16	{2-13,23-1,13-2}	
N17	{2-31,21-3,13-2}	
N18	{2-31,21-3,1-32}	
N19	{13-2,21-3,23-1}	
N20	{21-3,23-1,1-32}	
N21	{1-23,2-31,31-2}	
N22	{1-23,23-1,31-2}	
N23	{1-23,2-31,3-12}	
N24	{1-23,1-32,3-21}	
A1	{1-23,12-3,23-1}	
A2	{2-31,23-1,1-32}	
A3	{2-31,23-1,13-2}	
A4	{1-23,12-3,2-13}	
A5	{1-23,2-13,21-3}	$\{2^{n-1}\}_{n\geq 1}$
A6	{1-23,3-12,31-2}	
A7	{31-2,3-12,13-2}	
A8	{31-2,3-12,1-32}	
A9	{2-13,21-3,1-32}	
A10	{2-13,21-3,13-2}	
A11	{1-23,23-1,3-12}	$\{2n-2+1\}_{n\geq 1}$

Table 2.1 Permutations avoiding three patterns

symmetry class	avoided patterns	enumerating sequence
F1	$\{1-23, 2-13, 1-32\}$	
F2	$\{1-23, 2-13, 13-2\}$	
F3	$\{1-23, 21-3, 13-2\}$	
F4	$\{1-23, 13-2, 3-12\}$	$\{F_n\}_{n\geq 1}$
F5	$\{1-23, 1-32, 3-12\}$	
F6	$\{1-23, 1-32, 31-2\}$	
F7	$\{1-23, 13-2, 31-2\}$	
M1	$\{1-23, 12-3, 21-3\}$	
M2	$\{12-3, 21-3, 2-13\}$	$\{M_n\}_{n\geq 1}$
B1	$\{1-23, 21-3, 1-32\}$	$\{\binom{n}{\lceil n/2\rceil}\}_{n\geq 1}$
B2	$\{12-3, 1-23, 31-2\}$	
B3	$\{1-23, 2-31, 23-1\}$	$\{1+\binom{n}{2}\}$
C8	$\{12-3, 2-13, 32-1\}$	$\{3\}_{n\geq 3}$
C1	$\{1-23, 2-13, 3-21\}$	
C2	$\{1-23, 23-1, 32-1\}$	
C3	$\{1-23, 2-13, 32-1\}$	
C4	$\{1-23, 12-3, 3-21\}$	$\{0\}_{n\geq k}$
C5	$\{1-23, 21-3, 3-21\}$	
C6	$\{1-23, 21-3, 32-1\}$	
C7	$\{1-23, 2-31, 32-1\}$	

Table 2.2 Permutations avoiding three patterns

Enumerating sequence: $\{n\}_{n\geq1}$			
name	*avoided patterns*	*apply Proposition*	*to the symmetry class*
$d1$	$\{1-23, 2-13, 3-12, 21-3\}$	2.1.1	N1
$d2$	$\{1-23, 2-13, 31-2, 21-3\}$	2.1.1	N2
$d3$	$\{1-23, 2-13, 31-2, 3-12\}$	2.1.2	N2
$d4$	$\{1-23, 21-3, 31-2, 3-12\}$	2.1.2	N4
$d5$	$\{12-3, 3-12, 2-13, 21-3\}$	2.1.1	N5
$d6$	$\{12-3, 31-2, 2-13, 21-3\}$	2.1.1	N7
$d7$	$\{12-3, 31-2, 2-13, 3-12\}$	2.1.2	N7
$d8$	$\{12-3, 31-2, 21-3, 3-12\}$	2.1.2	N8
$d9$	$\{1-23, 2-13, 2-31, 21-3\}$	2.1.1	N9
$d10$	$\{1-23, 2-13, 2-31, 23-1\}$	2.1.3	N9
$d11$	$\{1-23, 2-13, 23-1, 21-3\}$	2.1.1	N10
$d12$	$\{1-23, 21-3, 2-31, 23-1\}$	2.1.3	N11
$d13$	$\{2-13, 2-31, 1-32, 21-3\}$	2.1.1	N13
$d14$	$\{2-13, 2-31, 1-32, 23-1\}$	2.1.3	N13
$d15$	$\{2-13, 23-1, 1-32, 21-3\}$	2.1.1	N14
$d16$	$\{2-13, 2-31, 13-2, 21-3\}$	2.1.1	N15
$d17$	$\{2-13, 2-31, 13-2, 23-1\}$	2.1.3	N15
$d18$	$\{2-13, 2-31, 13-2, 1-32\}$	2.1.4	N15
$d19$	$\{2-13, 23-1, 13-2, 21-3\}$	2.1.1	N16
$d20$	$\{2-13, 23-1, 13-2, 1-32\}$	2.1.4	N16
$d21$	$\{2-31, 21-3, 13-2, 23-1\}$	2.1.3	N17
$d22$	$\{2-31, 21-3, 13-2, 1-32\}$	2.1.4	N17
$d23$	$\{2-31, 21-3, 1-32, 23-1\}$	2.1.3	N18
$d24$	$\{13-2, 21-3, 23-1, 1-32\}$	2.1.4	N19
$d25$	$\{1-23, 2-31, 31-2, 23-1\}$	2.1.3	N21
$d26$	$\{1-23, 2-31, 31-2, 3-12\}$	2.1.2	N21
$d27$	$\{1-23, 23-1, 31-2, 3-12\}$	2.1.2	N22
$d28$	$\{1-23, 2-31, 3-12, 23-1\}$	2.1.3	N23
$d29$	$\{1-23, 2-13, 31-2, 12-3\}$	2.1.5	N2
$d30$	$\{1-23, 2-13, 3-12, 12-3\}$	2.1.5	N1
$d31$	$\{1-23, 2-13, 2-31, 12-3\}$	2.1.5	N9
$d32$	$\{1-23, 2-13, 23-1, 12-3\}$	2.1.5	N10
$d33$	$\{1-23, 21-3, 2-31, 12-3\}$	2.1.6	N11
$d34$	$\{1-23, 21-3, 23-1, 12-3\}$	2.1.6	N12
$d35$	$\{1-23, 21-3, 31-2, 12-3\}$	2.1.6	N4
$d36$	$\{1-23, 21-3, 3-12, 12-3\}$	2.1.6	N3
$d37$	$\{1-23, 2-31, 3-12, 12-3\}$	2.1.7	N23
$d38$	$\{1-23, 2-31, 31-2, 12-3\}$	2.1.7	N21

Table 2.3 Permutations avoiding four patterns

Enumerating sequence: $\{F_n\}_{n\geq 1}$			
name	*avoided patterns*	*apply Proposition*	*to the symmetry class*
$e1$	$\{1-23, 2-13, 1-32, 21-3\}$	2.1.1	F1
$e2$	$\{1-23, 2-13, 1-32, 12-3\}$	2.1.5	F1
$e3$	$\{1-23, 2-13, 13-2, 21-3\}$	2.1.1	F2
$e4$	$\{1-23, 2-13, 13-2, 1-32\}$	2.1.4	F2
$e5$	$\{1-23, 2-13, 13-2, 12-3\}$	2.1.5	F2
$e6$	$\{1-23, 21-3, 13-2, 1-32\}$	2.1.4	F3
$e7$	$\{1-23, 13-2, 3-12, 1-32\}$	2.1.4	F4
$e8$	$\{1-23, 1-32, 31-2, 3-12\}$	2.1.2	F6
$e9$	$\{1-23, 13-2, 31-2, 1-32\}$	2.1.4	F7
$e10$	$\{1-23, 13-2, 31-2, 3-12\}$	2.1.2	F7

Table 2.4 Permutations avoiding four patterns

Enumerating sequence: $\{2^{n-1}\}_{n\geq 1}$		
avoided patterns	*apply Proposition*	*to the symmetry class*
$\{1-23, 12-3, 2-13, 21-3\}$	2.1.1	A4
$\{31-2, 3-12, 13-2, 1-32\}$	2.1.4	A7
$\{2-13, 21-3, 13-2, 1-32\}$	2.1.4	A10
$\{2-31, 23-1, 1-32, 13-2\}$	2.1.4	A3

Table 2.5 Permutations avoiding four patterns

Enumerating sequence: $\{0\}_{n\geq k}$		
name	*choose a pattern from the following to add*	*to the symmetry class*
O1	$12-3$	$\{1-23, 2-13, 3-21\}$ (C1)
O2	$1-32$	
O3	$13-2$	
O4	$3-12$	
O5	$31-2$	
O6	$21-3$	
O7	$2-31$	
O8	$23-1$	
O9	$32-1$	
O10	$12-3$	$\{1-23, 23-1, 32-1\}$ (C2)
O11	$1-32$	
O12	$13-2$	
O13	$3-12$	
O14	$31-2$	
O15	$2-13$	
O16	$21-3$	
O17	$2-31$	
O18	$3-21$	
O19	$12-3$	$\{1-23, 2-13, 32-1\}$ (C3)
O20	$13-2$	
O21	$3-12$	
O22	$31-2$	
O23	$21-3$	
O24	$2-31$	
O25	$31-2$	$\{1-23, 12-3, 3-21\}$ (C4)
O26	$1-32$	
O27	$23-1$	
O28	$32-1$	
O29	$1-32$	$\{1-23, 21-3, 3-21\}$ (C5)
O30	$13-2$	
O31	$3-12$	
O32	$31-2$	
O33	$23-1$	
O34	$13-2$	$\{1-23, 21-3, 32-1\}$ (C6)
O35	$3-12$	
O36	$2-31$	
O37	$13-2$	$\{1-23, 2-31, 32-1\}$ (C7)

Table 2.6 Permutations avoiding four patterns

Enumerating sequence: $\{2\}_{n\geq 2}$			
1st pattern	*2nd pattern*	*3rd pattern*	*4th pattern*
$1-23$	$2-31$ *or* $23-1$	$1-32$ *or* $13-2$	$3-12$ *or* $31-2$
$1-23$	$2-13$ *or* $21-3$	$1-32$ *or* $13-2$	$3-12$ *or* $31-2$
$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$1-32$ *or* $13-2$	$3-12$ *or* $31-2$
$12-3$	$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$32-1$
$1-23$	$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$3-12$ *or* $31-2$
$1-23$	$2-13$ *or* $21-3$	$2-31$ *or* $23-1$	$1-32$ *or* $13-2$

Table 2.7 Permutations avoiding four patterns

Enumerating sequence: $\{0\}_{n\geq k}$	
choose a pattern from the following to add	*to the symmetry class*
$21-3,\ 2-31,\ 23-1,\ 1-32,\ 13-2,$ $3-12,\ 31-2,\ 32-1$	O1
$2-31,\ 23-1,\ 1-32,\ 13-2,\ 3-12,$ $31-2$	O6
$21-3,\ 2-31,\ 23-1,\ 1-32,\ 3-12,$ $31-2$	O19
$2-31,\ 23-1,\ 1-32,\ 13-2,\ 3-12$ $31-2$	O23
$1-32, 13-2, 3-12, 31-2,\ 32-1$	O8
$23-1,\ 1-32,\ 13-2,\ 31-2,\ 3-12$	O24
$12-3,\ 32-1,\ 13-2,\ 3-12,\ 31-2$	O29
$3-12,\ 13-2,\ 1-32,\ 23-1,\ 12-3$	O36
$1-32,\ 13-2,\ 3-12,\ 31-2$	O15
$13-2,\ 3-12,\ 31-2$	O2
$1-32,\ 2-31,\ 31-2$	O10
$12-3,\ 13-2,\ 3-12$	O32
$1-32,\ 13-2,\ 32-1$	O33
$3-21,\ 23-1,\ 1-32$	O34
$3-12,\ 31-2$	O3
$1-32,\ 13-2$	O7
$13-2,\ 3-12$	O9
$21-3,\ 13-2$	O11
$1-32,\ 3-12$	O20
$3-12,\ 23-1$	O26
$2-31,\ 32-1$	O27
$3-12$	O5
$2-31$	O12
$1-32$	O21
$3-12$	O30
$23-1$	O35

Table 2.8 Permutations avoiding five patterns

Enumerating sequence: $\{2\}_{n\geq k}$		
thanks to Proposition	*add the pattern*	*to the patterns*
2.1.1	$21-3$	$\{1-23, 2-13, 1-32, 3-12\}$ *or* $\{1-23, 2-13, 1-32, 31-2\}$ *or* $\{1-23, 2-13, 13-2, 3-12\}$ *or* $\{1-23, 2-13, 1-32, 31-2\}$
2.1.1	$21-3$	$\{1-23, 2-13, 2-31, 1-32\}$ *or* $\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 23-1, 1-32\}$ *or* $\{1-23, 2-13, 23-1, 13-2\}$
2.1.1	$21-3$	$\{1-23, 2-13, 2-31, 3-12\}$ *or* $\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 3-12\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
2.1.1	$21-3$	$\{1-23, 2-13, 23-1, 32-1\}$ *or* $\{1-23, 2-13, 2-31, 32-1\}$
2.1.1	$21-3$	$\{2-13, 2-31, 1-32, 3-12\}$ *or* $\{2-13, 2-31, 1-32, 31-2\}$ *or* $\{2-13, 2-31, 13-2, 31-2\}$
2.1.1	$21-3$	$\{2-13, 23-1, 1-32, 31-2\}$ *or* $\{2-13, 23-1, 1-32, 3-12\}$ *or* $\{2-13, 23-1, 13-2, 31-2\}$ *or* $\{2-13, 23-1, 13-2, 3-12\}$
2.1.2	$3-12$	$\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
2.1.2	$3-12$	$\{1-23, 2-13, 1-32, 31-2\}$ *or* $\{1-23, 2-13, 13-2, 31-2\}$
2.1.2	$3-12$	$\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
2.1.2	$3-12$	$\{1-23, 21-3, 1-32, 31-2\}$ *or* $\{1-23, 21-3, 13-2, 31-2\}$
2.1.2	$3-12$	$\{1-23, 2-31, 1-32, 31-2\}$ *or* $\{1-23, 2-31, 13-2, 31-2\}$
2.1.2	$3-12$	$\{1-23, 23-1, 1-32, 31-2\}$ *or* $\{1-23, 23-1, 13-2, 31-2\}$
2.1.3	$23-1$	$\{2-13, 2-31, 1-32, 31-2\}$
2.1.3	$23-1$	$\{1-23, 2-31, 13-2, 3-12\}$ *or* $\{1-23, 2-31, 13-2, 31-2\}$

Table 2.9 Permutations avoiding five patterns

Enumerating sequence: $\{2\}_{n\geq k}$		
thanks to Proposition	*add the pattern*	*to the patterns*
2.1.3	$23-1$	$\{1-23, 2-31, 1-32, 3-12\}$ *or* $\{1-23, 2-31, 1-32, 31-2\}$
2.1.3	$23-1$	$\{1-23, 21-3, 2-31, 1-32\}$ *or* $\{1-23, 21-3, 2-31, 13-2\}$ *or* $\{1-23, 21-3, 2-31, 3-12\}$ *or* $\{1-23, 21-3, 2-31, 31-2\}$
2.1.3	$23-1$	$\{1-23, 2-13, 2-31, 1-32\}$ *or* $\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 2-31, 3-12\}$ *or* $\{1-23, 2-13, 2-31, 31-2\}$
2.1.4	$1-32$	$\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 23-1, 13-2\}$
2.1.4	$1-32$	$\{1-23, 2-13, 13-2, 3-12\}$ *or* $\{1-23, 2-13, 13-2, 31-2\}$
2.1.4	$1-32$	$\{1-23, 21-3, 2-31, 13-2\}$ *or* $\{1-23, 21-3, 23-1, 13-2\}$
2.1.4	$1-32$	$\{1-23, 21-3, 13-2, 3-12\}$ *or* $\{1-23, 21-3, 13-2, 31-2\}$
2.1.4	$1-32$	$\{1-23, 2-31, 13-2, 3-12\}$ *or* $\{1-23, 2-31, 13-2, 31-2\}$
2.1.4	$1-32$	$\{1-23, 23-1, 13-2, 3-12\}$ *or* $\{1-23, 23-1, 13-2, 31-2\}$
2.1.5	$12-3$	$\{1-23, 2-13, 2-31, 1-32\}$ *or* $\{1-23, 2-13, 2-31, 13-2\}$ *or* $\{1-23, 2-13, 23-1, 1-32\}$ *or* $\{1-23, 2-13, 23-1, 13-2\}$
2.1.5	$12-3$	$\{1-23, 2-13, 2-31, 3-12\}$ *or* $\{1-23, 2-13, 2-31, 31-2\}$ *or* $\{1-23, 2-13, 23-1, 3-12\}$ *or* $\{1-23, 2-13, 23-1, 31-2\}$
2.1.5	$12-3$	$\{1-23, 2-13, 1-32, 3-12\}$ *or* $\{1-23, 2-13, 1-32, 31-2\}$
2.1.6	$12-3$	$\{1-23, 21-3, 23-1, 3-12\}$ *or* $\{1-23, 21-3, 23-1, 31-2\}$
2.1.6	$12-3$	$\{1-23, 21-3, 2-31, 1-32\}$ *or* $\{1-23, 21-3, 23-1, 1-32\}$
2.1.6	$12-3$	$\{1-23, 21-3, 2-31, 3-12\}$ *or* $\{1-23, 21-3, 2-31, 31-2\}$

Table 2.10 Permutations avoiding five patterns

Enumerating sequence: $\{n\}_{n\geq 1}$		
thanks to Proposition	*add the pattern*	*to the representative*
2.1.2	$3-12$	d2
2.1.3	$23-1$	d9
2.1.2	$3-12$	d6
2.1.2	$3-12$	d25
2.1.1	$21-3$	d14
2.1.1	$21-3$	d17
2.1.4	$1-32$	d16
2.1.1	$21-3$	d20
2.1.3	$23-1$	d18
2.1.4	$1-32$	d21
2.1.5	$12-3$	d9
2.1.5	$12-3$	d11
2.1.5	$12-3$	d1
2.1.5	$12-3$	d2
2.1.5	$12-3$	d10
2.1.5	$12-3$	d3
2.1.6	$12-3$	d12
2.1.6	$12-3$	d4
2.1.7	$12-3$	d28
2.1.7	$12-3$	d25

Table 2.11 Permutations avoiding five patterns

Enumerating sequence	*avoided patterns*	*apply Proposition*	*to the symmetry class*
$\{1+{n \atop 2}\}_{n\geq 1}$	$\{12-3, 1-23, 31-2, 3-12\}$	2.1.2	B2
$\{{n \atop \lceil n/2 \rceil}\}_{n\geq 1}$	$\{1-23, 21-3, 1-32, 12-3\}$	2.1.6	B1
$\{2^{n-2}+1\}_{n\geq 1}$	$\{1-23, 23-1, 3-12, 12-3\}$	2.1.8	A11
$\{3\}_{n\geq 3}$	$\{12-3, 2-13, 32-1, 21-3\}$	2.1.1	C8

Table 2.12 Permutations avoiding four patterns

Enumerating sequence: $\{F_n\}_{n\geq 1}$		
thanks to Proposition	*add the pattern*	*to the representative*
2.1.5	$12-3$	e1
2.1.5	$12-3$	e3
2.1.1	$21-3$	e4
2.1.3	$1-32$	e10

Table 2.13 Permutations avoiding five patterns

Chapter 3

A discrete continuity: from Fibonacci to Catalan

Fibonacci and Catalan numbers are very well known sequences. They appear in many combinatorial problems as they enumerate a great quantity of combinatorial objects. For instance, Fibonacci numbers are involved in the tiling of a strip, in rabbits' population growth, in bees' ancestors, ..., while Catalan numbers occur in the enumeration of several kinds of paths, trees, permutations, polyominoes and other combinatorial structures. Fibonacci numbers are described by the famous recurrence:

$$\begin{cases} F_0 = 1 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{cases}$$

from which the generating function:

$$F(x) = \frac{1}{1 - x - x^2}$$

arises, and the sequence begins with $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Catalan numbers have been deeply studied, too: they appear in many relations, also connected to other sequences or by themselves. They are defined by:

$$\begin{cases} C_0 = 1 \\ C_1 = 1 \\ C_n = \sum_{i=0}^{n-1} C_{n-1-i} C_i \end{cases}$$

The expression

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad \text{with } n \geq 0,$$

derived from the generating function

$$C(x) = \frac{1 + \sqrt{1-4x}}{2x},$$

is a closed formula for them and the sequence begins with the numbers $1, 1, 2, 5, 14, 42, 132, \ldots$.

Our question is: "What is there between Fibonacci and Catalan numbers?" For instance the following sequences:

- $\{c_n\}_{n\geq 0} = \{1, 1, 2, 4, 7, 13, 24, \ldots\}$, $(c_0 = 1, c_1 = 1, c_2 = 2, c_n = c_{n-1} + c_{n-2} + c_{n-3})$ Tribonacci numbers;
- $\{t_n\}_{n\geq 0} = \{1, 1, 2, 4, 8, 16, 32, \ldots, 2^{n-1}\}$, $(t_0 = 1, t_n = 2^{n-1})$;
- $\{p_n\}_{n\geq 0} = \{1, 1, 2, 5, 12, 29, 70, \ldots\}$, $(p_0 = 1, p_1 = 1, p_2 = 2, p_n = 2p_{n-1} + p_{n-2})$ Pell numbers;
- $\{\bar{F}_n\}_{n\geq 0} = \{1, 1, 2, 5, 13, 34, 89, \ldots\}$, $(\bar{F}_0 = 1, \bar{F}_1 = 1, \bar{F}_n = 3\bar{F}_{n-1} + \bar{F}_{n-2})$ even index Fibonacci numbers,

(for more details see the sequences M1074, M1129, M1413, M1439 in [Sl], respectively, where they are defined with different initial conditions) lay between Fibonacci and Catalan numbers (we call the last sequence *even* index Fibonacci numbers while other authors call them *odd* index Fibonacci numbers, but this depends on the initials conditions assumed for the Fibonacci sequence). We are looking for a unifying combinatorial interpretation for all these sequences, and others too. To this aim we will use permutations avoiding forbidden subsequences. The forbidden patterns used in this chapter are not generalized patterns.

The main idea we are going to base on, has already been used in [BDPP1] and [BDPP3]. Here, here we briefly recall that. It is well known that $|S_n(123, 213, 312)| = F_n$ and $|S_n(123)| = C_n$, as mentioned in the abstract. The patterns 213 and 312, which are not present in the second equality, can be seen as particular cases of more general patterns. More precisely, 213 can be obtained from the pattern $r_k = k(k-1)(k-2)\dots 21(k+1)$ with $k = 2$, while 312 is the pattern $q_k = 1(k+1)k(k-1)\dots 21$ with $k = 2$, again. When k grows, the patterns r_k and q_k increase their length, then in the limit (k grows to ∞) they can be not considered in the enumeration of the permutations π of $S_n(123, r_k, q_k)$ since, for each $n \geq 0$, any π does not surely contain a pattern of infinite length. In other words, starting from the case $k = 2$ (involving Fibonacci numbers), for each $k > 2$ we provide a class of pattern avoiding permutations where the pattern are suitably generalized in order to make them "disappear" when k grows, leading to the class $S(123)$ enumerated by the Catalan numbers. We say that there is a sort of "continuity" between Fibonacci and Catalan numbers since we provide a succession of generating functions $\{g_k(x)\}_{k\geq 2}$ with $g_2(x) = F(x)$ and whose limit is $C(x)$.

As a matter of fact, in this chapter this aim is reached in two steps: first only the pattern 312 is generalized so that we arrive to the class $S(123, 213)$ enumerated by $\{2^{n-1}\}$, then the pattern 213 is increased in order to obtain the class $S(123)$. Nevertheless it is possible to make "disappear" both the patterns at the same time obtaining similar results.

As mentioned above, the forbidden patterns used in this chapter are not generalized patterns. Nevertheless, we think that the approach we are going to use could produce similar results about a continuity between different remarkable (or not) sequences.

3.1 Preliminaries

Permutations avoiding forbidden subsequences have been widely studied by many authors [BK, BDPP2, BDPP5, Che, Chu, EM1, Gi, Gu, Kra, Krem, SS, St, W1, W2, W4]. A very efficient and natural method to enumerate classes of permutations was proposed by Chung et al. [Chu] and Rogers [Ro], and, later, by West [W1]. It consists in generating permutations in S_n from permutations in S_{n-1} by inserting n in all the positions such that a forbidden subsequence does not arise (we denote these positions by a '$\diamond$'). These positions are the *active sites*, while a *site* is any position between two consecutive elements in a permutation or the position before the first element or after the last one. If a permutation in $S_{n-1}(\Gamma_1, \ldots, \Gamma_j)$, where Γ_i's are forbidden patterns, contains k active sites, then it generates k permutations in $S_n(\Gamma_1, \ldots, \Gamma_j)$. In the sequel, we denote the i-th active site as the site located before π_i.

In order to show how we can enumerate classes of permutations by this method, we consider the class $S_n(123)$. Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation in $S_n(123)$ such that $\pi_1 > \pi_2 > \cdots > \pi_{k-1} < \pi_k$. Then the first k sites are active, since the insertion of $n+1$ in one of these positions does not create a subsequence of kind 123. On the contrary, the sites on the right of π_k are not active because the insertion of $n+1$ produces the subsequence $\pi_{k-1}\pi_k(n+1)$ which is of kind 123. Therefore, from the permutation

$\diamond\pi_1 \diamond \pi_2 \diamond \cdots \diamond \pi_{k-1} \diamond \pi_k\pi_{k+1} \cdots \pi_n$

we obtain the following ones:

$\diamond(n+1)\pi_1 \diamond \pi_2 \diamond \cdots \diamond \pi_{k-1} \diamond \pi_k \pi_{k+1} \cdots \pi_n$

$\diamond\pi_1 \diamond (n+1)\pi_2 \cdots\cdots \pi_n$

$\diamond\pi_1 \diamond \pi_2 \diamond (n+1)\pi_3 \cdots \pi_n$

$\vdots$

$\diamond\pi_1 \diamond \pi_2 \diamond \cdots \diamond \pi_{k-1} \diamond (n+1)\pi_k \cdots \pi_n$

which have respectively $(k+1), 2, 3, \ldots, k$ active sites. We remark that from a permutation π having k active sites we obtain k permutations having $(k+1), 2, 3, \ldots, k$ active sites, independently from the length of the permutation. Such a permutation is labelled with (k). We can "condense" this property into the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(k+1) \quad . \end{cases} \tag{3.1}$$

The label (1), the axiom of the succession rule, is the number of active sites of the empty permutation ε which is the only permutation with length $n = 0$, meaning that ε generates the minimal permutation $\pi = 1$ with length $n = 1$. In turn, $\pi = \diamond 1 \diamond$ has two active sites, then it produces two permutations: this fact is described by the second line of the rule $(1) \rightsquigarrow (2)$ (the production of the axiom).

Usually, from a succession rule we can obtain a functional equation or a system of equations from which one can obtain the generating function $f(x) = \sum_{n\geq 0} a_n x^n$ where a_n is the number of objects on level n. From the above example for $S(123)$, it is possible to obtain (we omit the calculus) the generating function $C(x)$ for Catalan numbers. Moreover, $|S_n(123)| = C_n$, for $n \geq 0$.

The enumeration of the permutations of $S_n(123, 132, 213)$ is also briefly illustrated, which is the starting point of our argument. In the permutations of this class only the first two sites can be active: the insertion of $n+1$ in another site would produce the subsequence $\pi_1\pi_2(n+1)$ which is of kind 123 or 213. If $\pi_1 < \pi_2$ then only the first site is active because the insertion of $n+1$ in the second site would produce the subsequence $\pi_1\pi_2(n+1)$ which is of kind 132. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $S_n(123, 132, 213)$; if $\pi_1 < \pi_2$, from $\diamond\pi_1\pi_2\cdots\pi_n$ we obtain $\diamond(n+1)\diamond\pi_1\pi_2\cdots\pi_n$ which has two active sites; if $\pi_1 > \pi_2$, from $\diamond\pi_1\diamond\pi_2\cdots\pi_n$ we obtain $\diamond(n+1)\diamond\pi_1\pi_2\cdots\pi_n$ and $\diamond\pi_1(n+1)\pi_2\cdots\pi_n$ having two and one active sites, respectively. This construction can be encoded by the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2) \end{cases} \tag{3.2}$$

The above succession rule is an example of *finite* succession rule since only a limited number of different labels appear in it. It is easily seen that it leads to Fibonacci numbers and $|S_n(123, 213, 312)| = F_n$, for $n \geq 0$.

In the last part of this section, we only note that the permutations of the class $S(123, 213)$, which is the intermediate step between the above considered classes (see the Introduction), have exactly two active sites (the first two sites), so that the corresponding succession rule is

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)(2) \end{cases} \tag{3.3}$$

It is easy to prove that the related enumerating sequence $\{t_n\}_{n\geq 0}$ is defined by

$$\begin{cases} t_0 = 1 \\ t_n = 2^{n-1}, \quad n \geq 1 \end{cases}$$

and $|S_n(123, 213)| = t_n$. The corresponding generating function is $t(x) = \frac{1-x}{1-2x}$. In the sequel, we refer to this sequence simply with $\{2^{n-1}\}_{n\geq 0}$.

We conclude by observing that all the considered sequences take into account the empty permutation which is enumerated by C_0, F_0 and t_0. Moreover, in each presented succession rule the axiom refers to it and the production $(1) \rightsquigarrow (2)$ describes its behavior.

3.2 From Fibonacci to 2^{n-1}

Consider a permutation $\pi \in S_n(123, 213, 1(k+1)k\dots 2)$. His structure is essentially known thanks to [EM], where the authors analyze the permutations of $S_n(123, 132, k(k-1)\dots 21(k+1))$ which is equivalent to the class we are considering (the permutations of the former are the reverse complement of the latter). In the same paper the authors show that those permutations are enumerated by the sequence of k-generalized Fibonacci numbers, providing also the related generating function. Here, we give an alternative proof of the same facts by using the ECO method [BDPP1]. To this aim, we recall the structure of the permutations referring directly to the class $S_n(123, 213, 1(k+1)k\dots 2)$, nevertheless we omit the easy proofs that one can recover from [EM].

If $\pi \in S_n(123, 213, 1(k+1)k\dots 2)$, then:

- either $\pi_1 = n$ or $\pi_2 = n$;
- if $\pi_1 = n$, then $\pi = n\tau$, with $\tau \in S_{n-1}(123, 213, 1(k+1)k\dots 2)$;
- if $\pi_2 = n$, then $\pi_1 = n - j$, with $j \in \{1, 2, \dots, k-1\}$, and $\pi = (n-j)n(n-1)\dots(n-j+1)\sigma$, with $\sigma \in S_{n-j-2}(123, 213, 1(k+1)k\dots 2)$.

If $\pi \in S_n(123, 213, 1(k+1)k\dots 2)$, denote $\pi^{(i)}$ the permutations such that $\pi_1 = n-i$. The active sites of π are the first two sites: the insertion of $n+1$ in any other site would create the forbidden pattern 123 or 213. More precisely, the permutations $\pi^{(j)}$ with $j \in \{0, 1, 2, \dots, k-2\}$ have label (2) (the first two sites are active), while $\pi^{(k-1)}$ has label (1) (the first site is active). The son of the permutation $\pi^{(k-1)}$ is the permutation of $S_{n+1}(123, 213, 1(k+1)k \dots 2)$ obtained from π by inserting $n+1$ in its first active site, which we denote $\bar{\pi}^{(0)}$. It is easily seen that $\bar{\pi}^{(0)}$ has, in turn, label (2). The two sons of the permutations with label (2) are $\bar{\pi}^{(0)}$ and $\bar{\pi}^{(j+1)}$ ($\bar{\pi}^{(j+1)}$ is obtained from π by inserting $n+1$ in the second active site). Therefore, all these permutations have, in turn, label (2) but $\bar{\pi}^{(k-1)}$ which has label (1). Since all the labels (2) have not the same production, it is suitable to label each permutation $\pi^{(j)}$ ($j \in \{0, 1, 2, \dots, k-2\}$) with (2_j) in order to recognize the permutation $\pi^{(k-2)}$ whose sons have labels (1) and (2). Then, the above description can be encoded by:

$$\begin{cases} (1) & \\ (1) & \rightsquigarrow \quad (2_0) \\ (2_j) & \rightsquigarrow \quad (2_0)(2_{j+1}), \qquad \text{for} \quad j = 0, 1, 2, \dots, k-3 \\ (2_{k-2}) & \rightsquigarrow \quad (2_0)(1) \end{cases}$$

We now deduce the generating function $T^{(k)}(x, y)$ of the permutations of $S(123, 213, 1(k+1)k \dots 2)$, according to their length and number of active sites. To this aim we consider the subsets T_1 of the permutations with label (1) and T_{2_j}, with $j = 0, 1, 2, \dots, k-2$, of the permutations with label (2_j). It is obvious that these subsets form a partition of $S(123, 213, 1(k+1)k \dots 2)$. Denote with $T_1(x, y) = \sum_{\pi \in T_1} x^{n(\pi)} y^{f(\pi)}$ the generating function of T_1 and $T_{2_j}(x, y) = \sum_{\pi \in T_{2_j}} x^{n(\pi)} y^{f(\pi)}$ the generating function of T_{2_j} ($j = 0, 1, \dots, k-2$), where $n(\pi)$ and $f(\pi)$ are the length and the number of active sites of a permutation π, respectively. From the above succession rule the following system is derived:

$$\begin{cases} T_1(x,y) &= y + xy\sum_{\pi\in T_{2_{k-2}}} x^{n(\pi)} \\ \\ T_{2_0}(x,y) &= xy^2(T_1(x,1)+\sum_{i=0}^{k-2} T_{2_i}(x,1)) \\ \\ T_{2_j}(x,y) &= xy^2 T_{2_{j-1}}(x,1), \quad j=1,2,\dots,k-2 \end{cases} .$$

Clearly, it is $T^{(k)}(x,y) = T_1(x,y) + \sum_{j=0}^{k-2} T_{2_j}(x,y)$ and, if $y=1$, $T^{(k)}(x,1)$ is the generating function of the permutations of $S(123,213,1(k+1)k\dots2)$ according to their length. From the above system (we omit the calculus), it follows:

$$T^{(k)}(x,1) = \frac{1-x}{1-2x+x^{k+1}} \qquad .$$

Note that if k grows to ∞, the generating function $t(x)$ related to the sequence $\{2^{n-1}\}_{n\geq 0}$ (enumerating the permutations of $S(123,213)$, see Section 4.1.1) is obtained. For each $k \geq 2$, we get an expression which is the generating function of the k-generalized Fibonacci numbers. For $k=1$, the formula leads to $\frac{1}{1-x}$ which is the generating function of the sequence $\{1\}_{n\geq0}$ enumerating the permutations of $S_n(123,213,12) = S_n(12) = n\ (n-1)\ \dots 2\ 1$. For $k=3$ the succession is

$$\begin{cases} (1) & & \\ (1) & \rightsquigarrow & (2_0) \\ (2_0) & \rightsquigarrow & (2_0)(2_1) \\ (2_1) & \rightsquigarrow & (2_0)(1) \end{cases} ,$$

which defines the Tribonacci numbers, whose generatring function is $T^{(3)}(x,1) = \frac{1}{1-x-x^2-x^3}$.

3.3 From 2^{n-1} to Catalan

Let π be a permutation of $S_n(123, k(k-1)\dots21(k+1))$. Then if $\pi_i = n$ it is $i \in \{1,2,\dots,k\}$, otherwise if $\pi_j = n$ with $j \geq k+1$, it should be

$\pi_1 > \pi_2 > \ldots > \pi_k$ in order to avoid the pattern 123. But in this way the entries $\pi_1, \pi_2, \ldots, \pi_k, \pi_j$ are a pattern $k(k-1)\ldots 21(k+1)$ which is forbidden.

If α_π denotes the minimum index j such that $\pi_{j-1} < \pi_j$, we can describe the active sites of π by using α_π.

1. If $\alpha_\pi = j \leq k$, then the active sites are the first j sites of π. The insertion of $n+1$ in any other site would create the pattern 123. In this case π as label (j).

2. If $\alpha_\pi > k$, then the active sites of π are the first k sites since the insertion of $n+1$ in any other site would lead to the occurrence of the forbidden patterns $k(k-1)\ldots 21(k+1)$ or 123. In this case π has label (k).

In order to describe the labels of the sons of π, in the sequel we denote $\bar{\pi}^{(i)}$ the permutation $\bar{\pi} \in S_{n+1}(123, k(k-1)\ldots 21(k+1))$ obtained from π by inserting $n+1$ in the i-th active site of π.

1. If π has label (k), it is not difficult to see that $\alpha_{\bar{\pi}^{(1)}} = \alpha_\pi + 1 > k$, then $\bar{\pi}^{(1)}$ has label (k) again. While, if we consider $\bar{\pi}^{(i)}$, with $i = 2, 3, \ldots, k$, then $\alpha_{\bar{\pi}^{(i)}} = i$ and $\bar{\pi}^{(i)}$ has label (i). Therefore the production of the label (k) is $(k) \rightsquigarrow (2)(3)\ldots(k)(k)$.

2. If π has label (j) with $j \in \{2, 3, \ldots, k-1\}$, then it is easily seen that $\alpha_{\bar{\pi}^{(1)}} = \alpha_\pi + 1 \leq k$ and $\bar{\pi}^{(1)}$ has label $(j+1)$ (note that in this case $\alpha_\pi = j$). While if we consider $\bar{\pi}^{(i)}$, with $i = 2, 3, \ldots, j$, then $\alpha_{\bar{\pi}^{(i)}} = i$ and $\bar{\pi}^{(i)}$ has label (i). Therefore the production of (j) is $(j) \rightsquigarrow (2)(3)\ldots(j)(j+1)$.

The above construction can be encoded by the succession rule:

$$\begin{cases} (1) & & \\ (1) & \rightsquigarrow & (2) \\ (j) & \rightsquigarrow & (2)(3)\dots(j)(j+1), \qquad \text{for } \; j = 2,3,\dots,k-1 \\ (k) & \rightsquigarrow & (2)(3)\dots(k)(k) \qquad , \end{cases}$$

where the axiom and its production refer to the empty permutation generating the permutation $\pi = 1$, which, in turn, produces two sons: $\pi = 12$ and $\pi = 21$. Using the theory developed in [DFR1], the production matrix related to the above succession rule is

$$P_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 2 \end{pmatrix} \qquad ,$$

with k rows and columns. For each $k \geq 2$, it is easy to see that the matrix P_k can be obtained from P_{k-1} as follows:

$$P_k = \begin{pmatrix} 0 & u^T \\ 0 & P_{k-1} + eu^T \end{pmatrix} \quad ,$$

where u^T is the row vector $(1, 0, \dots, 0)$ and e is the column vector $(1, 1, \dots, 1)^T$ (both $k-1$-dimensional). If $f_{P_k}(x)$ is the generating function according to the length of the permutations associated to P_k, from a result in [DFR1] (more precisely Proposition 3.10), the following functional equation holds:

$$f_{P_k}(x) = \frac{1}{1 - x f_{P_{k-1}}(x)} \; .$$

In the limit, we have $f(x) = \frac{1}{1-xf(x)}$ which is the functional equation verified by the generating function of the Catalan numbers $C(x)$.
As a particular case, it is possible to check that for $k = 3$, the sequence of the even index Fibonacci numbers is involved. The obtained succession rule is

$$\begin{cases} (1) & & \\ (1) & \rightsquigarrow & (2) \\ (2) & \rightsquigarrow & (2)(3) \\ (3) & \rightsquigarrow & (2)(3)(3) \end{cases},$$

leading to the related generating function $\bar{F}(x) = \frac{1-2x}{1-3x+x^2}$.

3.4 Another way for achieving the same goal

In Section 3.2, starting from $S(123, 213, 132)$ and using the knowledge that $S(123, 213)$ is enumerated by $\{2^{n-1}\}_{n\geq 0}$, the pattern 132 has been generalized in $1(k + 1)k \dots 2$, in order to make it "disappear". Since the class $S(123, 132)$ is enumerated by $\{2^{n-1}\}_{n\geq 0}$, too, one can choose the pattern 213 instead of 132 (among the forbidden patterns of the permutations of $S(123, 213, 132)$) as the one to be generalized. Indeed, there is no a particular reason why we chose the pattern 132 to make it disappear.

Similarly, starting from $S(123, 132)$ and recalling that $|S_n(p)| = C_n \; \forall \, p \in S_3$, either the pattern 123 or the pattern 132 can be generalized in order to find a class enumerated by the Catalan numbers.

The difference between a choice with respect to another one lies in the fact that different ECO construction for the permutations are expected. Therefore, different succession rules for the same sequence could be found.

3.4.1 From Fibonacci to 2^{n-1}

Starting from $S(123, 213, 132)$, here we generalize the pattern 213 considering the class $S(123, 132, k(k-1)\dots 21(k+1))$, for $k \geq 3$. This class has already been described in [EM], where the author provides the structure of its permutations. From his results, it is possible to deduce the following succession rule (similarly to Section 3.2, the details are omitted), encoding the construction of those permutations:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (1)^{h-1}(h+1) \quad \text{for } h < k \\ (k) \rightsquigarrow (1)^{k-1}(k) \end{cases}$$

In [EM] the author shows also that the k-generalized Fibonacci numbers are the enumerating sequence of the permutations of $S(123, 132, k(k-1)\dots 21(k+1))$. This fact can be derived also by solving the system that can be obtained from the above succession rule, with a technique similar to that one used in Section 3.2 leading to the same generating function $T^k(x, 1) = \frac{1-x}{1-2x+x^{k+1}}$. This agrees with the fact that in the limit for $k \to \infty$, the class to be considered is $S(123, 132)$, enumerated by $\{2^{n-1}\}_{n\geq 0}$ [SS]. We note that it is possible to describe the permutations of $S(123, 132)$ with the succession rule

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (1)^{h-1}(h+1) \end{cases},$$

from which one can get that the related generating function is, again, $t(x) = \frac{1-x}{1-2x}$.

The particular case $k = 3$ is marked: the obtained succession rule is

$$\begin{cases} (1) & \\ (1) & \rightsquigarrow \quad (2) \\ (2) & \rightsquigarrow \quad (1)(3) \\ (3) & \rightsquigarrow \quad (1)(1)(3) \end{cases} .$$

corresponding to the sequence of Tribonacci numbers, as one can check by deriving the related generating function $T^3(x, 1) = \frac{1}{1-x-x^2-x^3}$.

3.4.2 From 2^{n-1} to Catalan

Starting from $S(123, 132)$, the pattern 132 is generalized in $(k-1)(k-2)\dots21(k+1)k$, with $k \geq 3$. Moreover, the construction of the permutations of $S(123, (k-1)(k-2)\dots21(k+1)k)$ is described and the corresponding succession rule is showed. Finally, we prove that the corresponding generating function is, in the limit for $k \to \infty$, the generating function of the Catalan numbers $C(x)$.

Let π be a permutation of $S_n(123, (k-1)(k-2)\dots21(k+1)k)$. We denote:

- $r = \min\{1, 2, \dots, n\}$ such that $\pi_{r-1} < \pi_r$;
- $s = \min\{1, 2, \dots, n\}$ and $t = \min\{1, 2, \dots, n\}$ such that, fore some indexes $m_1 < m_2 < \dots < m_{k-2} < s < t$, it is $\pi_{m_1}\pi_{m_2}\dots\pi_{m_{k-2}}\pi_s\pi_t \simeq (k-1)(k-2)\dots21k$ (the two subsequences are order-isomorphic and π_s and π_t correspond to the 1 and to the k of the pattern $(k-1)(k-2)\dots21k$);
- $\alpha_\pi = \min\{r, s\}$;
- $\bar{\pi}^{(l)}$ the permutation of $S_{n+1}(123, (k-1)(k-2)\dots21(k+1)k)$ obtained from π by inserting $n+1$ in the l-th site.

We prove that π has α_π active sites which are the first α_π sites of π.

It is easily seen that that the insertion of $n+1$ in any site among the first α_π sites of π, does not induce either the pattern 123 or the pattern $(k-1)(k-2)\dots 21(k+1)k$. On the other hand, if $\alpha_\pi = r$, then the insertion of $n+1$ in the l-th site, $l > \alpha_\pi$, would create the pattern 123 in the entries $\bar\pi^{(l)}_{r-1}\bar\pi^{(l)}_r\bar\pi^{(l)}_l$. While, if $\alpha_\pi = s$, then the insertion of $n+1$ in the i-th site, $\alpha_\pi + 1 \le i \le t$, would create the pattern $(k-1)(k-2)\dots 21(k+1)k$ in the entries $\bar\pi^{(i)}_{m_1}\bar\pi^{(i)}_{m_2}\dots\bar\pi^{(i)}_{m_{k-2}}\bar\pi^{(i)}_{\alpha_\pi}\bar\pi^{(i)}_i\bar\pi^{(i)}_{t+1}$ (recall that $\bar\pi^{(i)}_i = n+1$ and $\bar\pi^{(i)}_{t+1} = \pi_t$). Finally, if $i \ge t+1$, the pattern 123 would appear in the entries $\bar\pi^{(i)}_{\alpha_\pi}\bar\pi^{(i)}_t\bar\pi^{(i)}_i$.

Denote (h) the label of π, whit $h = \alpha_\pi$. In order to describe the labels of the sons $\bar\pi^{(l)}$, $l = 1, 2, \dots, h$, of π, we have:

1. If $h < k$ (note that on this case $\alpha_\pi = r$ or, if $\alpha_\pi = s$, then $s = k-1$), then the permutation $\bar\pi^{(1)} = (n+1)\pi_1\pi_2\dots\pi_{\alpha_\pi}\dots\pi_k\dots\pi_n$, so that $\alpha_{\bar\pi^{(1)}} = \alpha_\pi + 1$. Therefore $\bar\pi^{(1)}$ has label $(h+1)$. While if we consider the permutations $\bar\pi^{(j)}$, $j = 2, 3, \dots, h$, it is $\alpha_{\bar\pi^{(j)}} = j$ since $\bar\pi^{(j)}_{j-1} < \bar\pi^{(j)}_j (= n+1)$. So $\bar\pi^{(j)}$ has label (j) and we conclude that the production of (h) is $(h) \rightsquigarrow (2)(3)\dots(h)(h+1)$.

2. If $h \ge k$, then $\bar\pi^{(1)} = (n+1)\pi_1\pi_2\dots\pi_k\dots\pi_{\alpha_\pi}\dots\pi_n$, so that $\alpha_{\bar\pi^{(1)}} = \alpha_\pi + 1$. Therefore $\bar\pi^{(1)}$ has label $(h+1)$. Note that in both cases $\alpha_\pi = r$ or $\alpha_\pi = s$ it is $\pi_1 > \pi_2 > \dots > \pi_{\alpha_\pi - 1}$. Then, if we consider the permutations $\bar\pi^{(j)}$, $j = k, k+1, \dots, \alpha_\pi$, we obtain $\alpha_{\bar\pi^{(j)}} = k-1$, regardless of j, since $\bar\pi^{(j)}_1\bar\pi^{(j)}_2\dots\bar\pi^{(j)}_{k-1}\bar\pi^{(j)}_j \simeq (k-1)(k-2)\dots 1k$. Then $\bar\pi^{(j)}$ has label $(k-1)$, for $j = k, k+1, \dots, \alpha_\pi$. For the remaining sons $\bar\pi^{(j)}$, $j = 2, 3, \dots, k-1$, it is easily seen that $\bar\pi^{(j)}_{j-1} < \bar\pi^{(j)}_j (= n+1)$. So, $\bar\pi^{(j)}$ has label (j). We conclude that, in this second case, the production of (h) is $(h) \rightsquigarrow (2)(3)\dots(k-2)(k-1)^{h-k+2}(h+1)$.

The above description of the generation of the permutations of $S(123, (k-1)(k-2)\dots 21(k+1)k)$ can be then encoded in the following succession rule

Ω_k:

$$\Omega_k = \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)\cdots(h)(h+1) \quad \text{for } h < k \\ (h) \rightsquigarrow (2)\cdots(k-2)(k-1)^{h-k+2}(h+1) \quad \text{for} \quad h \geq k \end{cases} .$$

For $k = 2$, the class $S(123, 132)$ is obtained, whose corresponding succession rule has been considered in Section 3.4.1. Note that it does not correspond with the one obtained from the above one poising $k = 2$.

For $k = 3$ (the class is $S(123, 2143)$) we get the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)^{h-1}(h+1) \end{cases} ,$$

leading to the even index Fibonacci numbers. Note that it is different from the succession rule corresponding to the same numbers of Section 3.3. Its associated production matrix [DFR1] is:

$$M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & \cdots \\ 0 & 3 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

For each $k \geq 4$, it is easy to check that the production matrix related to Ω_k satisfies

$$M_k = \begin{pmatrix} 0 & u^T \\ 0 & M_{k-1} + eu^T \end{pmatrix} ,$$

where $u^T = (1, 0, 0, \ldots)$ and $e = (1, 1, 1, \ldots)^T$. Then, if $g_{M_k}(x)$ is the corresponding generating function, we deduce [DFR1]:

$$g_{M_k}(x) = \frac{1}{1 - x g_{M_{k-1}(x)}} .$$

If $g(x)$ denotes the limit of $g_{M_k}(x)$, the functional equation $g(x) = \frac{1}{1-xg(x)}$ is obtained, which is verified by the generating function $C(x)$ of the Catalan numbers.

3.5 From Fibonacci to Catalan directly

This section summarizes the results found when the two patterns 132 and 213 are generalized at the same time, considering the class $S(123,(k-1)(k-2)\dots 21(k+1)k), k(k-1)\dots 21(k+1))$ in order to obtain the class $S(123)$, when k grows to ∞. Most of the proofs are omitted but they can easily recovered by the reader. At the first step, for $k=3$, we find the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(2)(3) \end{cases}$$

corresponding to $S_n(123, 2143, 3214)$. This class is enumerated by Pell numbers which we define with the recurrence:

$$\begin{cases} p_0 = 1 \\ p_1 = 1 \\ p_2 = 2 \\ p_n = 2p_{n-1} + p_{n-2}, \quad n \geq 3 \end{cases}$$

Note that the initial conditions are different from the usual ones (which are $p_0 = 0$ and $p_1 = 1$) in order to consider the empty permutation ε, for $n = 0$.

For a general k we have the class $S_n(123,(k-1)\cdots 1(k+1)k, k(k-1)\cdots 1(k+1))$. We briefly describe the construction of the permutations of the class (the details are omitted). Let π be a permutation of the class. It is easily seen that if $\pi_l = n$, then $l \leq k$. Therefore, if (h) denotes the label of π, it is $h \in \{1, 2, \dots, k\}$. Now, if $h < k$, then $\bar{\pi}^{(1)}$ has label $(h+1)$ and $\bar{\pi}^{(j)}$, $j = 2, 3, \dots, h$, has label (j). While, if $h = k$, then $\bar{\pi}^{(1)}$ has label (k), $\bar{\pi}^{(j)}$, $j = 2, 3, \dots, k-1$, has label (j) and $\bar{\pi}^{(k)}$ has label $(k-1)$, again. The construction can be encoded in the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)(3)\cdots(h-1)(h)(h+1) \ h < k \\ (k) \rightsquigarrow (2)(3)\cdots(k-1)(k-1)(k) \quad . \end{cases}$$

For each k, considering the associated production matrices [DFR1] and the corresponding generating functions, it possible to prove that, in the limit, the generating function of the Catalan numbers is obtained.

3.5.1 A continuity between Pell numbers and even index Fibonacci numbers

We conclude by showing that it is possible to find a "continuity" between Pell and even index Fibonacci numbers. We start from the class $S_n(123, 2143, 3214)$ (obtained by posing $k = 3$ in the preceding succession rule) enumerated by Pell numbers, then we generalize the pattern 2143, so obtaining the classes $S(123, 3214, 21(k+1)k(k-1)\ldots 43)$.

Let $\pi \in S_n(123, 3214, 21(k+1)k(k-1)\ldots 43)$. Then, if $\pi_l = n$, it is $l \leq 3$ in order to avoid the patterns 123 and 3214. Therefore, π has at most 3 active sites (the first three sites of π). We denote r_π the number of entries of π with index $j \geq 3$ such that $\pi_j > \pi_1$ (note that if $\pi_1 > \pi_2$, then $r_\pi = 0$). It is:

- $\pi_{j_1} > \pi_{j_2} > \ldots > \pi_{j_{r_\pi}}$ (the pattern 123 is forbidden);
- $r_\pi \leq (k-2)$ (the pattern $21(k+1)k\ldots 43$ is forbidden);
- the elements π_{j_i} are adjacent in π in order to avoid 123 or $21(k+1)k\ldots 43$.

If π starts with an ascent (i.e. $\pi_1 < \pi_2$), then only the first two sites are active, since the insertion of $n+1$ in any other site would create the pattern 123: the permutation π has label (2).

If π starts with a descent (i. e. $\pi_1 > \pi_2$), then the number of its active sites depends on r_π:

1. If $r_\pi = h < k-2$, then π has three active sites. Let (3_h) be its label. The permutation $\bar{\pi}^{(1)}$ (obtained by π by inserting $n+1$ in the first site) starts with a descent and $r_{\bar{\pi}^{(1)}} = 0$ (since $\bar{\pi}^(1)_1 = n+1$); therefore, $\bar{\pi}^{(1)}$ has label (3_0). The son $\bar{\pi}^{(2)}$ starts with an ascent and its label is (2). The last son $\bar{\pi}^{(3)}$ starts with a descent and $r_{\bar{\pi}^{(3)}} = h+1$, so its label is (3_{h+1}). The production of (3_h) is $(3_h) \rightsquigarrow (2)(3_0)(3_{h+1})$.

2. If $r_\pi = k-2$, then π has two active sites, since the insertion in the third site would create the pattern $21(k+1)k\ldots43$, while the insertion in any other site surely creates the pattern 123. Its son $\bar{\pi}^{(1)}$ has label (3_0) since it starts with a descent and $r_{\bar{\pi}^{(1)}} = 0$. While the other son $\bar{\pi}^{(2)}$ starts with an ascent and has label (2). Therefore, the production of label (2) is $(2) \rightsquigarrow (2)(3_0)$.

The following succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)(3_0) \\ (3_j) \rightsquigarrow (2)(3_0)(3_{j+1}), \quad \text{for} \quad j = 0,1,2,\ldots,k-3 \\ (3_{k-3}) \rightsquigarrow (2)(2)(3_0) \end{cases}$$

summarizes the construction of the class $S(123, 3214, 21(k+1)k\ldots43)$. Solving the system one can deduce from the above rule, the generating function $\bar{F}_k(x) = \frac{1-2x+x^k}{1-3x+x^2+x^k}$ is obtained, which in the limit is the generating function of the even index Fibonacci numbers $\bar{F}(x)$.

Starting from the class $S(123, 2143, 3214,)$, one can generalize the pattern 3214 instead of 2143. The class we get is $S(123, 2143, k(k-1)\ldots32(k+1)1)$ and the succession rule describing its construction is (the easy proof is omitted):

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)^{h-1}(h+1) \quad \text{for} \quad h < k \\ (k) \rightsquigarrow (2)^{k-1}(k) \quad . \end{cases}$$

Once again, one can prove that the corresponding generating function is $\bar{F}_k(x)$, leading, in the limit, to $\bar{F}(x)$.

3.6 Remarks

In order to summarize the several "continuities" we have here proposed, we condense our results in Figure 3.1 where a straight line represents a direct step and a dashed line represents a family of permutations obtained by generalizing one or two patterns.

The results we found for permutations can be easily extended to Dyck paths and planar trees by means of ECO method [BDPP, BDPP1]. We can find classes of paths and trees described by the finite succession rules we introduced by imposing some conditions on the height of paths and the level of their valleys and on the outdegree and level of nodes in the trees.

Figure 3.1 allows to see the different three ways we have followed to describe a discrete "continuity" between Fibonacci and Catalan numbers: the generalization of a single pattern (the rightmost and the leftmost path from the top to the bottom in the figure) and the generalization of a pair of patterns (central path in the figure). In particular, following the rightmost and the leftmost path in the graph, the intermediate level of the permutations enumerated by $\{2^{n-1}\}_{n\geq 0}$ is encountered. For each k, our approach produces two different class of permutations enumerated by the same sequence, indeed the two corresponding generating functions are the same for each k. We note that, in this way, we can provide two different succession

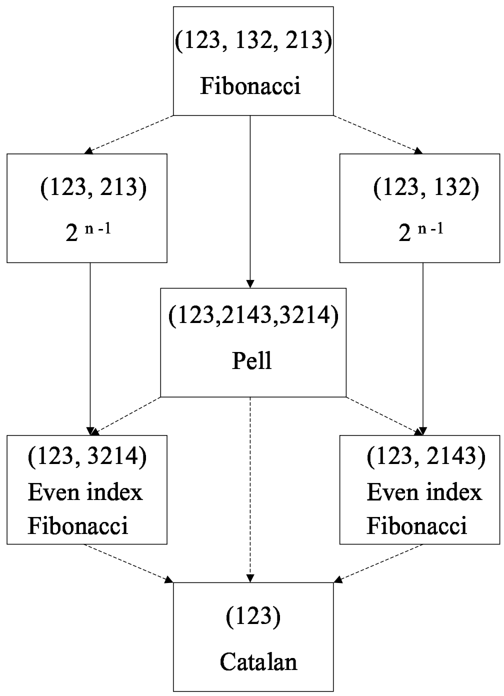


Figure 3.1 The graph of permutations.

rules encoding the same sequence. An instance can be seen by looking at the succession rules the reader can find at the end of the Sections 3.2 and 3.4.1.

The same happens with the succession rule at the end of Section 3.4.2 and the succession rule of the particular case ($k = 3$) of Section 3.4.2, which encode the sequence of the even index Fibonacci numbers. Really, we did not prove that this is the case for each k related to the classes of permutations used to describe the discrete continuity between $\{2^{n-1}\}_{n\geq 0}$ and Catalan numbers, since we did not get the explicit formulas of the generating functions.

Chapter 4

Order properties of pattern avoiding permutations

It is well known that the symmetric group S_n endowed with the strong Bruhat order does not posses a lattice structure. The main results of this chapter concern certain subsets of S_n of (generalized) pattern avoiding permutations which are proved to have a distributive lattice structure. The leading idea is considering some remarkable lattice paths (more precisely Dyck, Motzkin and Schröder paths) on which it is possible to define a natural order [FP2] such that those set of paths are distributive lattices. Then, via some suitable known bijections, certain classes of corresponding restricted permutations are considered, analyzing which are the properties of the induced order. In the cases of Dyck and Motzkin paths this order coincides with the induced strong Bruhat order of the symmetric group. We note that similar results were found by other authors [BW, Dr], nevertheless they were concerned with the weak order on permutations.

The covering relation in the strong Bruhat order is as follows. For $\sigma, \tau \in S_n$, the permutation τ covers σ if it can be obtained by σ interchanging two entries σ_i and σ_j ($i < j$ and $\sigma_i < \sigma_j$), such that $\sigma_l < \sigma_i$ or $\sigma_l > \sigma_j$, for each l such that $i < l < j$. In the weak order, τ covers σ if it is obtained by σ

interchanging two adjacent elements and if it has more inversions than σ. Moreover, we recall that in both orders the rank function is given by the number of inversions.

4.1 A distributive lattice structure connecting Dyck paths noncrossing partitions and 312-avoiding permutations

We start by considering noncrossing partitions. A set partition is said to be *noncrossing* when, given four elements, $1 \leq a < b < c < d \leq n$, such that a, c are in the same block and b, d are in the same block, then the two blocks coincide. The set of all noncrossing partitions of an n-set will be denoted $NC(n)$. We make use of the following notation: each noncrossing partition $\pi = B_1|B_2 \cdots |B_k$ is expressed by listing its blocks B_i in increasing order of their maxima, whereas the elements inside each block are listed in decreasing order. It is clear that every (noncrossing) partition can be uniquely written in this way, which will be called here the *standard notation* for (noncrossing) partitions.

It is known that noncrossing partitions can be endowed with the refinement order, so to obtain the partition lattices, first studied by Kreweras [Krew], which have many interesting properties. Nevertheless, they are not distributive. Our question is if there is the possibility of defining some *interesting* distributive lattice structure on noncrossing partitions? We claim that the answer is affirmative by explicitly finding an order on noncrossing partitions which is isomorphic to at least two combinatorially meaningful distributive lattices.

We first consider Dyck paths and define an order on them as follows: given two Dyck paths P, Q of the same length, we say that $P \leq Q$ when P entirely lies below Q (possibly coinciding with Q in some points). It is possible to show [FP2] that the set of Dyck paths of any given length endowed with this order is a distributive lattice. Our idea is to transfer such a structure on noncrossing partitions along a famous bijection (see [Si]). We have called *Bruhat noncrossing partition lattices* the distributive lattices of noncrossing partitions arising in this way; Section 4.1.3 is devoted to the study of some properties of these lattices. Moreover, Bruhat noncrossing partition lattices turn out to be isomorphic to an even more interesting class of lattices. It is not difficult to explicitly find a bijection between noncrossing partitions and 312-avoiding permutations. More precisely, we show that such a bijection is an order-isomorphism between the Bruhat lattice of noncrossing partitions of an n set and the class $S_n(312)$ of 312-avoiding permutations of an n set endowed with the (strong) Bruhat order. As a byproduct, we have that $S_n(312)$ is a distributive sublattice of the symmetric group of order n with the Bruhat order. These results are contained in Section 4.1.4, where we also find a criterion to determine the meet and the join of two permutations in $S_n(312)$. To the best of our knowledge, the only paper dealing with this kind of matters is [P], where the author determines the Bruhat posets (arising from Weyl groups) which are lattices. However, the language and the aims of [P] are totally different from the ones of our approach. It would be interesting to compare our results with those of Proctor. However, it seems to us that our result is the first one concerning the order structure induced by the strong Bruhat order on a class of pattern-avoiding permutations.

The final part of this introduction is devoted to the explanation of the main notation we use through this section and to the presentation of the basics of some general theories we refer to in the next pages.

The covering relation of any poset we are going to consider throughout the section will be denoted by the symbol $\prec$, so that $x \prec y$ means "x is covered by y". The set (and the lattice) of partitions of $[n] = \{1, 2, \ldots, n\}$

will be denoted by $\Pi(n)$. If $\pi \in \Pi(n)$, we will always use the notation $\pi = B_1|B_2|\ldots|B_k$, where the B_i's are the blocks of π, the elements inside each block are in decreasing order and $\max B_i < \max B_j$, for $i < j$. We will often deal with Dyck paths and, depending on the context, we will find convenient to describe them in several different ways. Therefore a Dyck path will be alternatively described as a particular lattice path in the discrete plane $\mathbf{N} \times \mathbf{N}$ (and denoted by capital letters like $P, Q, R, \ldots$) or as a function $f : \mathbf{N} \longrightarrow \mathbf{N}$ satisfying certain properties (and denoted by lowercase letters like $f, g, h, \ldots$) or else as a particular word on the two-letter alphabet $\{U, D\}$ (and denoted by Greek letters such as $\omega(U, D), \psi(U, D), \ldots$). We leave to the reader the details of the descriptions of Dyck paths we have sketched in the previous sentence.

4.1.1 Preliminaries on set partitions

We start by recalling the main properties of set partitions with respect to the classical order by refinement. Therefore, this section has to be intended as a selected survey of some properties of partition lattices. In order not to repeat the content of some classical textbook word by word, we have chosen to use an alternative language. What we have obtained is a presentation of some classical results on set partitions in the framework of the ECO method and succession rules, which we hope to be of some interest to the reader.

Given $\pi, \rho \in \Pi(n)$, define $\pi \leq \rho$ when every block of π is contained into some block of ρ. The many properties of this classical order can be found in several sources, such as [A1, S1]. Here we only mention that $\Pi(n)$ endowed with this *refinement order* is a lattice which is neither distributive nor modular. Nevertheless, it possesses a rank function: the rank of $\pi = B_1|B_2|\ldots|B_k$ is $n - k$. The Whitney numbers of the partition lattices are the well-known Stirling numbers of the second kind. The Hasse diagram of $\Pi(4)$ is shown in Figure 4.1.

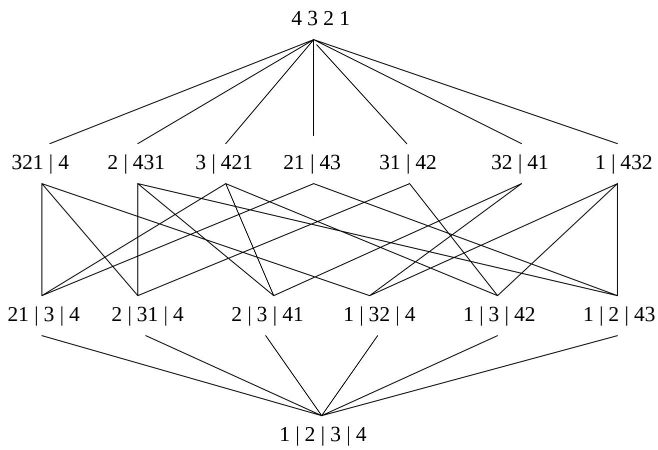


Figure 4.1 $\Pi(4)$.

A classical recursive construction of set partitions works as follows: given $\pi = B_1|B_2|\ldots|B_k \in \Pi(n)$, either add $n+1$ to any of the blocks of π or insert the new block $B_{k+1} = \{n+1\}$. In this way we obtain $k+1$ new partitions of $[n+1]$, namely $\pi_i = B_1|\ldots|B_{i-1}|B_{i+1}|\ldots|B_k|(n+1)B_i$, $i = 1,\ldots,k$ and $\pi_{k+1} = B_1|\ldots|B_k|(n+1)$. Observe that this classical construction can be interpreted into the framework of the ECO method and encoded by the following succession rule:

$$\begin{cases} (1) \\ (k) \rightsquigarrow (k)^{k-1}(k+1). \end{cases}$$

This succession rule should be read as follows. Label each partition with the number of its blocks, plus 1. Then a partition labelled (k) produces k new partitions; of these, $k-1$ are still labelled (k), whereas the last one is labelled $(k+1)$.

We say that $\pi, \rho \in \Pi(n)$ are *ECO-equivalent* when they are produced by the same partition of $\Pi(n-1)$ (i.e., they have the same father in the generating tree of the above succession rule). It turns out that each equivalence class with the induced order is a *flat* [DP], that is, a poset with minimum

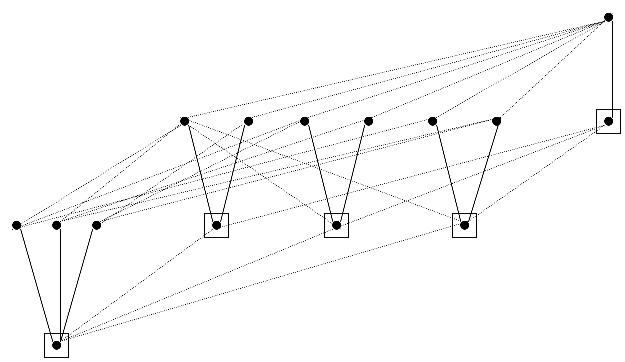


Figure 4.2 The flat partition of $\Pi(4)$.

in which all the remaining elements are maximal. Therefore, thanks to the previous ECO-construction, we get a "flat partition" of every $\Pi(n)$.

The following proposition, whose proof is straightforward, contains some properties of this flat partition.

Proposition 4.1.1 *The lattice $\Pi(n)$ is partitioned into $\mathcal{B}_{n-1}$ flats*[1]*; more precisely, for $1 \leq k \leq n-1$, there are $\mathcal{S}_{n-1,k}$ flats* [2] *of cardinality $n+1-k$, whose minima have rank $k-1$. The order induced on the set of such minima is isomorphic to $\Pi(n-1)$. If we compute the Whitney numbers of $\Pi(n)$ using this flat partition we obtain the well-known recursion for the $\mathcal{S}_{n,k}$'s, namely $\mathcal{S}_{n+1,k} = \mathcal{S}_{n,k-1} + k\mathcal{S}_{n,k}$.*

4.1.2 Noncrossing partitions and Dyck paths

A partition of $1, 2, \dots, n$ is *noncrossing* when, given four elements, $1 \leq a < b < c < d \leq n$, such that a, c are in the same block and b, d are in the same block, then the two blocks coincide. The set of all noncrossing partitions of an n-set will be denoted $NC(n)$. We refer the reader to the fairly complete

[1] $\mathcal{B}_n$ is the n-th Bell number.

[2] $\mathcal{S}_{n,k}$ is the n, k entry of the triangle of the Stirling numbers of the second kind.

survey [Si] and to the references therein for the plentiful applications of this notion.

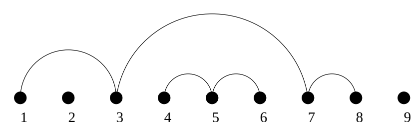


Figure 4.3 The noncrossing partition $2|654|8731|9 \in NC(9)$.

The refinement order can be restricted to noncrossing partitions: what we obtain is again a lattice, which is usually referred to as the *noncrossing partition lattice*. Among the main features of these lattices we recall here that they are not distributive and the lattice operations are different from those of the partition lattices (the join of two noncrossing partitions needs not be noncrossing within the full partition lattice).

Noncrossing partitions are enumerated by Catalan numbers, so, as it often happens, it is possible to find a bijection with Dyck paths. The nice bijection we are going to describe can also be found, for instance, in [De, Si]. Fix a Dyck path and label its up steps by enumerating them from left to right (so that the k-th up step is labelled k). Next assign to each down step the same label of the up step it is matched with. Now consider the partition whose blocks are constituted by the labels of each sequence of consecutive down steps. Such a partition is easily seen to be noncrossing. In Figure 4.4 we have illustrated this bijection on a concrete example; the bold labels next to the down steps are the elements of the corresponding noncrossing partition, whereas the up steps are simply labelled in increasing order.

Now denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$. It is possible to define a natural order on $\mathcal{D}_n$ by setting $f \leq g$ whenever $f(i) \leq g(i)$, for every $i \in \mathbf{N}$. This means that $f \leq g$ when f "lies weakly" below g. The set $\mathcal{D}_n$, endowed with such an order, turns out to be a distributive lattice, which has been studied in some detail in [FP2] under the name of *Dyck lattice (of*

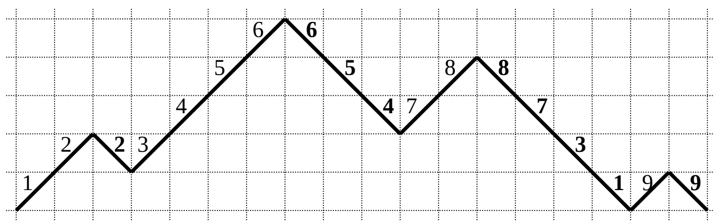


Figure 4.4 The Dyck path associated with 2|654|8731|9.

order n). We point out that Dyck lattices have also been considered in [CJ], where the authors speak of *geometric inclusion* of paths.

Our idea is to transfer the order structure of Dyck lattices along the above described bijection. In this way we define a new order on noncrossing partitions. The distributive lattices obtained in this way will be called *Bruhat noncrossing partition lattices.* The reason of this name, which is at present rather obscure, will become clear in Section 4.1.4. Our main goal is to give a satisfactory description of such lattices.

In the rest of this section we propose a presumably new construction for noncrossing partitions by transferring a well-known ECO-construction of Dyck paths[3] described in [BDPP1] along the previous bijection. As a byproduct, we will find a statistic on noncrossing partitions whose distribution is given by the ballot numbers.

Fix a partition $\pi = B_1|\ldots|B_{k-1}|B_k \in NC(n)$, with $B_k = a_1 \ldots a_r$. Starting from π we construct $r+1$ new partitions, namely $\pi_1 = B_1|\ldots|B_k|(n+1)$, $\pi_i = B_1|\ldots|B_{k-1}|a_1 \ldots a_{r-i+1}|(n+1)a_{r-i+2}\ldots a_r$ (for $i = 2,\ldots r$), and $\pi_{r+1} = B_1|\ldots|B_{k-1}|(n+1)a_1 \ldots a_r$. This construction of noncrossing par-

[3]The construction goes as follows: a Dyck path P of length $2n$ generates a set of Dyck paths of length $2n+2$ by inserting a peak in each point of the last sequence of consecutive down steps.

tition is "isomorphic" to the above mentioned ECO-construction for Dyck paths. The next proposition translates on noncrossing partitions some enumerative results produced by this construction.

Proposition 4.1.2 *Starting from $NC(n-1)$, every partition of $NC(n)$ is obtained precisely once by means of the above construction. More precisely, if a noncrossing partition is labelled by the cardinality of the block containing its maximum, plus 1 (so that $\pi = B_1|\ldots|B_s$ is labelled $(|B_s|+1)$), then our construction can be described by means of the following succession rule:*

$$\Omega : \begin{cases} (1) \\ (k) \rightsquigarrow (2)(3)(4)\cdots(k)(k+1) \end{cases} .$$

Consequently, we have that the number of noncrossing partitions of $[n]$ such that the block containing n has cardinality $k-1$ equals the number of Dyck paths of length $2n$ whose last descent has $k-1$ down steps, which is the ballot number $\frac{k+1}{n}\binom{2n-k-2}{n-k-1}$ (see [BDPP1] for the enumerative combinatorics of the rule Ω).

Remark. We point out that the above construction of noncrossing partitions has many similarities with the one given in the last section of [A], where the author illustrates the basics of a method of enumeration via ballot tables. It would be interesting to relate this approach with the ECO methodology.

4.1.3 The Bruhat noncrossing partition lattice

In the rest of this section it is tacitly assumed that noncrossing partitions are endowed with the Bruhat order defined above.

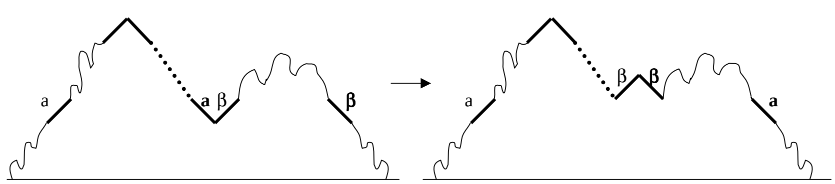
Given two noncrossing partitions π, ρ we look for some condition to recognize if $\pi \prec \rho$ or not. The following theorem gives a precise answer to this problem.

Theorem 4.1.1 *(Characterization of coverings) Given two noncrossing partitions $\pi, \rho \in NC(n)$, we have $\pi \prec \rho$ if and only if ρ is obtained from π by moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ and either*

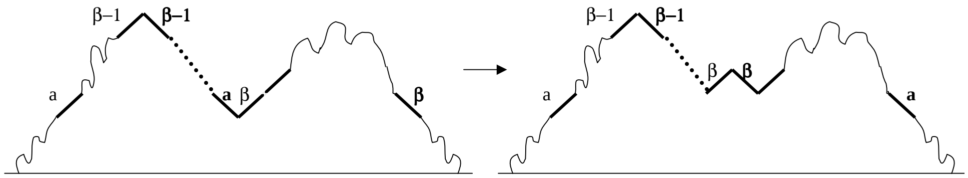
1. *keeping β inside $\tilde{B}$, if $\beta = \max \tilde{B}$, or*

2. *adding a new block $\overline{B} = \{\beta\}$, if $\beta \neq \max \tilde{B}$.*

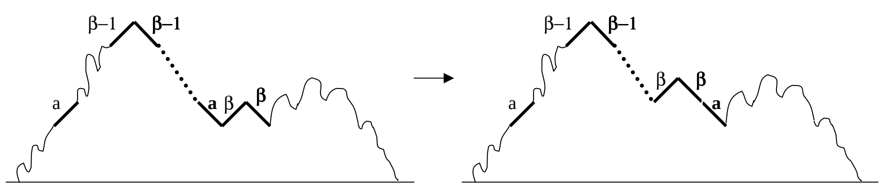
Proof. Suppose that P_π, P_ρ are the Dyck paths associated with π, ρ, respectively. The fact that $P_\pi \prec P_\rho$ in $\mathcal{D}_n$ means that P_ρ is obtained from P_π by replacing a valley with a peak. In the context of noncrossing partitions this amounts to moving the minimum a of a block, since the down step of a valley is the last step of a descent. The element a is moved into the block containing the element corresponding to the down step matched with the up step of the valley. It follows directly from the above bijection that such a down step has label equal to $\beta = \max B + 1$, where B is the block containing a in π. The following figure illustrates these facts.



Now, what happens with the element β? There are essentially two different cases. If the up step of the valley in P_π is followed by another up step, then β is not the maximum of its block in π, and it is easy to check that in ρ it becomes a singleton block (since in P_ρ the corresponding step is preceded and followed by up steps).



If the up step of the valley is followed by a down step, then β is the maximum of its block in π, and it remains in the same block also in ρ, as illustrated in the next figure.



□

Example. Given the partition $2|54|631 \in NC(6)$, there are precisely two partitions covering it, which are $3|54|621$ (2 is moved and 3 is not the maximum of its block) and $2|5|6431$ (4 is moved and 6 is the maximum of its block).

It is interesting to observe that the two "instructions" 1. and 2. in the previous theorem have a striking analogy with the definition of a *filler* given in [DS]. Recall that a point $i \in \{2, 3, \ldots, n\}$ is called a filler of $\pi \in NC(n)$ if either (i) $i-1$ and i are in the same block and i is the largest element of its block, or (ii) i forms a singleton block and $i-1$ is not the largest element in its block. Indeed, a filler is produced whenever a valley preceded by an up step is changed into a peak in the associated Dyck path. Thus a filler in a noncrossing partition corresponds to a down step preceded by a long ascent in the associated Dyck path (where a long ascent is a sequence of two or more consecutive up steps). Therefore, the number of noncrossing partitions of an n-set having k fillers coincides with the number $T_{n,k}$ of Dyck paths of length $2n$ having k long ascents, namely (see [S1]):

$$T_{n,k} = \frac{1}{n+1}\binom{n+1}{k}\sum_{j=0}^{n-2k}\binom{k+j-1}{k-1}\binom{n+1-k}{n-2k-j}.$$

Our next result is a criterion to compare two given noncrossing partitions. In order to properly state it, we need to introduce a technical definition. We define the *max-vector* of a noncrossing partition $\pi \in NC(n)$ to be the vector $\max(\pi) = (\mu_1, \dots, \mu_n)$ such that μ_i is the maximum of the first i elements of π. So, for instance, if $\pi = 2|31|54$, then $\max(\pi) = (2, 3, 3, 5, 5)$. We invite the reader to check that the max-vector uniquely determines its associated noncrossing partition. This fact will be very important in the sequel.

Theorem 4.1.2 *(Characterization of the Bruhat order of NC) Let $\pi, \rho \in NC(n)$. Then $\pi \le \rho$ if and only if $\max(\pi) \le \max(\rho)$ in the coordinatewise order.*

Proof. Let $\omega_1 = \omega_1(U, D)$ and $\omega_2 = \omega_2(U, D)$ be the two Dyck paths corresponding to π and ρ, respectively. Then it is clear that $\omega_1 \le \omega_2$ if and only if every prefix of ω_1 contains at least as many D's as the corresponding prefix of ω_2. This can be translated on partitions using max-vectors. Indeed, if $\max(\pi) = (\mu_1, \dots, \mu_n)$ and $\max(\rho) = (\nu_1, \dots, \nu_n)$, consider the two vectors $(\overline{\mu_1}, \dots, \overline{\mu_n})$ and $(\overline{\nu_1}, \dots, \overline{\nu_n})$, where $\overline{\mu_i} = \mu_i + i$ and $\overline{\nu_i} = \nu_i + i$. Then, it is not difficult to observe that $\overline{\mu_i}$ and $\overline{\nu_i}$ encode the position of the i-th D in the corresponding Dyck path. From the hypotheses, we have that the i-th

D of ω_1 occurs before the i-th D of ω_2, and so $\overline{\mu_i} \leq \overline{\nu_i}$. Since this holds for every $i \leq n$, the thesis follows.

□

Example. Let $\pi = 2|43|51|6$, $\rho = 43|52|61 \in NC(6)$. We easily find $\max(\pi) = (2, 4, 4, 5, 5, 6)$ and $\max(\rho) = (4, 4, 5, 5, 6, 6)$. It is immediate to see that $\max(\pi) \leq \max(\rho)$, whence $\pi \leq \rho$.

Remark. Observe that, if $\pi \prec \rho$, then $\max(\pi)$ and $\max(\rho)$ differ precisely in one position.

At this stage it is worth observing that the Bruhat noncrossing partition lattices can be alternatively described using *increasing parking functions*. Recall that an increasing parking function is a sequence $a_1 \leq a_2 \leq \cdots \leq a_n$ such that $a_i \leq i$, for all $i \leq n$. The poset of increasing parking functions of $[n]$ with the coordinatewise order is clearly a distributive lattice. It is not difficult to show that such a lattice is isomorphic to the dual Bruhat noncrossing partition lattice on n elements. Indeed, the correspondence mapping $(a_1, a_2 \ldots, a_{n-1}, a_n)$ into the sequence $(n+1-a_n, n+1-a_{n-1}, \ldots, n+1-a_2, n+1-a_1$ is an order-reversing bijection between increasing parking functions of $[n]$ and the set of max-vectors of noncrossing partitions of $[n]$.

It is known [FP2] that Dyck lattices possess a rank function (simply because they are distributive lattices) which is essentially given by the *area* bounded by a Dyck path and the x-axis. More precisely, if $A(P)$ is the area of a Dyck path P of length n, then the rank of P inside its Dyck lattice is given by $r(P) = \frac{A(P)-n}{2}$. Our next goal is to translate the parameter "area under Dyck paths" into a parameter on noncrossing partitions, in order to define a rank on the Bruhat noncrossing partition lattices.

Our first result is a formula for the area of Dyck paths in terms of its peaks and valleys. Since we have not found such a formula in the literature, we also propose a proof for the reader's convenience.

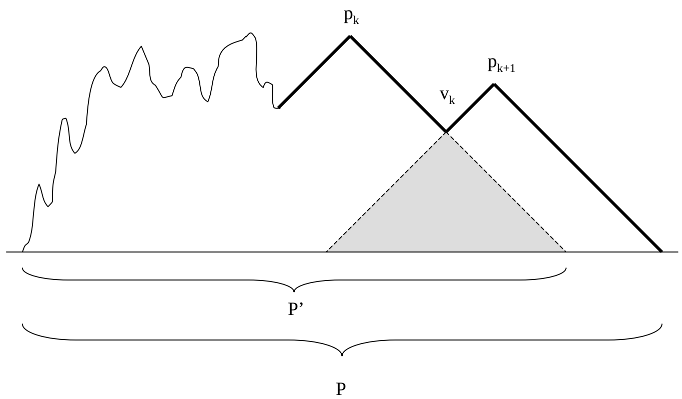


Figure 4.5 How P' is obtained from P.

Lemma 4.1.1 *Let P be a Dyck path. Let p_i and v_j denote the height of the i-th peak and the j-th valley of P, respectively. Assuming by convention that, if P has k peaks, then $v_k = 0$, we have:*

$$A(P) = \sum_i (p_i^2 - v_i^2). \tag{4.1}$$

Proof. We proceed by induction on the number of peaks. If a Dyck path P has only one peak, then it is the maximum of its Dyck lattice, and the formula immediately follows. Now suppose that P has $k+1$ peaks. Consider the path P' obtained by P by removing the last peak, i.e. coinciding with P up to the k-th peak and then ending with a sequence of down steps (see Figure 4.5).

It is now easy to see that

$$A(P) = A(P') + p_{k+1}^2 - v_k^2,$$

whence, thanks to the induction hypothesis:

$$A(P) = \sum_i (p_i^2 - v_i^2). \quad \square$$

Now we are ready to find a formula to express the rank of a partition in the Bruhat noncrossing partition lattice.

Theorem 4.1.3 *$NC(n)$ is a distributive lattice, and therefore it is ranked. More precisely, if $\pi = B_1|\dots|B_k \in NC(n)$, then its rank is given by:*

$$r_n(\pi) = \frac{A(\pi) - n}{2}, \tag{4.2}$$

where

$$A(\pi) = \sum_{i=1}^{k} \left(|B_i| \left(2b_i - 2\sum_{j=1}^{i-1} |B_j| - |B_i| \right) \right) \tag{4.3}$$

(here $b_i = \max B_i$).

Proof. Let P be a Dyck path and $\pi = B_1|\dots|B_k$ its associated noncrossing partition. Clearly the number of peaks of P coincides with the number of blocks of π. If (x, p_i) is the i-th peak of P, then p_i is the difference between the number u of up steps and the number d of down steps up to (x, p_i). In the language of noncrossing partitions, u is the maximum b_i of the i-th block of π, and d is simply the sum of the cardinalities of the first $i-1$ blocks of π. Therefore we have:

$$p_i = b_i - \sum_{j=1}^{i-1} |B_j|.$$

Analogously, if v_i is the height of the i-th valley of P, we find:

$$v_i = b_i - \sum_{j=1}^{i} |B_j|.$$

Plugging these quantities in the formula found in Lemma 4.1.1, we finally obtain:

$$\begin{aligned} A(\pi) &= \sum_{i=1}^{k} \left(\left(b_i - \sum_{j=1}^{i-1} |B_j| \right)^2 - \left(b_i - \sum_{j=1}^{i} |B_j| \right)^2 \right) \\ &= \sum_{i=1}^{k} \left(|B_i| \left(2b_i - 2\sum_{j=1}^{i-1} |B_j| - |B_i| \right) \right). \quad \square \end{aligned}$$

4.1.4 Relationship with the strong Bruhat order on permutations

The last formula given for the rank of a noncrossing partition inside its Bruhat lattice is not as easy to understand as the rank function for Dyck paths. In order to find a better way to express this parameter, we make use of the concept of (generalized) pattern avoiding permutation. What we obtain is yet another description of Bruhat noncrossing partition lattices which provides some important information on the (strong) Bruhat order of the symmetric groups.

Proposition 4.1.3 *Removing the bars in noncrossing partitions defines a bijection between $NC(n)$ and the set $S_n(312)$ of 312-avoiding permutations of $[n]$, for any $n \in \mathbf{N}$.*

Proof. First observe that, for any $n \in \mathbf{N}$, $S_n(312) = S_n(31\text{-}2)$, since it is known that these two finite sets are both enumerated by Catalan numbers and obviously $S_n(312) \subseteq S_n(31\text{-}2)$. Now, if a pattern 31-2 appears in a noncrossing partition, then, denoting by $b < c < a$ the three elements corresponding to such a pattern, a and b must belong to the same block, and the maximum d of the block containing c must be larger than a (since the maximum of a block in a noncrossing partition is larger than every element preceding it). Thus, the four elements a, b, c, d would constitute a crossing, against the hypothesis.

□

Remark. In the rest of this section we will make an extensive use of the above described canonical bijection. In particular, we will freely switch from a noncrossing partition to its associated 312-avoiding permutation without stating it explicitly. Moreover, we will always use the same Greek letters $(\pi, \rho, \sigma, \ldots)$ to denote both a noncrossing partition and its associated 312-avoiding permutation. Finally, observe that each maximum of a block of a noncrossing partition corresponds to a left-to-right maximum in the corresponding permutation, that is an element which is greater than every other element on its left.

Observe that the composition of the bijection between Dyck paths and noncrossing partitions with the above one between noncrossing partitions and 312-avoiding permutations is precisely the bijection considered in [BK] and in [F]. It also appears in [Kra], as a bijection between 132-avoiding permutations and Dyck paths. Moreover, we would like to point out that a simple visualization of this bijection, which involves lattice paths connecting opposite corners of the permutation array, is given in [EP, Re]. Using such a description, some of the properties of the Bruhat order on noncrossing partitions, such as Theorem 4.2.3, can be suitably rephrased.

Among the features of above mentioned bijection, a very interesting one is stated in [BK], where the authors show that the area of a Dyck path corresponds to the inversion number of the associated permutation. Since the rank function of the strong Bruhat order on permutations is precisely the inversion number, we are led to conjecture a close relation between our noncrossing partition lattices and the subposets induced by the Bruhat order on 312-avoiding permutations. The next theorem shows the truth of our conjecture.

Theorem 4.1.4 *Let* $(S_n(312); \leq)$ *be the poset obtained by transferring the structure of the Bruhat noncrossing partition lattice* $NC(n)$ *along the previous bijection. This is precisely the subposet induced on* $S_n(312)$ *by the strong Bruhat order of the symmetric group* S_n*. Therefore* $S_n(312)$ *is a distributive sublattice of* S_n *endowed with the strong Bruhat order.*

Proof. What we have to show is that the Hasse diagram of the Bruhat noncrossing partition lattice is isomorphic to that of $S_n(312)$ with the induced strong Bruhat order. To do this, it is enough to prove that the sets of elements covering a noncrossing partition and its associated 312-avoiding permutation coincide, via the bar-removing bijection.

Let π, ρ be noncrossing partitions, and suppose that $\pi \prec \rho$ in the Bruhat noncrossing partition lattice. This means that ρ is obtained from π by using one of the two rules described in Theorem 4.2.5. In both cases, the permutation ρ is obtained from the permutation π by interchanging the minimum a of a block B with $\beta = \max B + 1$. On permutations this means that the inversion number of ρ is larger than that of π (since $a < \beta$). Now to conclude that $\pi \prec \rho$ in $S_n(312)$ it remains only to show that the above transposition does not generate other inversions, or, equivalently, that all the entries between a and β in π are either smaller than a or larger than β. Indeed, $\beta - 1$ is the maximum of B, so it appears before a in π. Hence, if there is an element x such that $a < x < \beta$ and x is between a and β in π, then we would have a pattern 312, which is excluded. Therefore we have shown that, if $\pi \prec \rho$ in $NC(n)$, then also $\pi \prec \rho$ in $S_n(312)$.

To conclude the proof we will show that, if $\pi \prec \rho$ in $S_n(312)$, then necessarily ρ is obtained from π as in Theorem 4.2.5. From the hypothesis it follows that ρ differs from π by a transposition of a pair of elements a and β. Suppose that $a < \beta$ and so a appears before β in π. If a was not a minimum in the noncrossing partition associated with π, then there would be an entry $x < a$ appearing after a, and so in ρ the elements β, x, a would show a pattern 312. Therefore a must be the minimum of its block B in the noncrossing partition π. Now set $b = \max B$. We claim that $\beta = b + 1$. Indeed, if not, then $\beta - 1$ would not appear between a and β in π (since otherwise ρ would contain too many inversions). Clearly $\beta - 1$ cannot appear before b either, since every entry before b must be smaller than b. Thus $\beta - 1$ lies necessarily on the right of β in π. But in this case the permutation ρ would contain a pattern 312 in the entries $\beta, a, \beta - 1$, a contradiction. Therefore $\beta = b + 1$, and the theorem is finally proved.

□

At this stage it is worth mentioning the following, remarkable corollary.

Corollary 4.1.1 *For any $n \in \mathbf{N}$, the Dyck lattice $\mathcal{D}_n$ is isomorphic to the lattice $S_n(312)$ with the strong Bruhat order.*

Our next goal is to find a synthetic description of the meet and join operations in the Bruhat lattices of 312-avoiding permutations.

Let $\pi = \pi_1 \cdots \pi_n,\ \rho = \rho_1 \cdots \rho_n \in S_n(312)$. Define the permutation $\pi \vee \rho = \sigma_1 \cdots \sigma_n$ by setting σ_i equal to the largest element among those smaller than or equal to $\max\{\pi_1, \dots, \pi_i, \rho_1, \dots, \rho_i\}$ not yet appeared in the previous positions. Analogously, the permutation $\pi \wedge \rho = \tau_1 \cdots \tau_n$ is defined by setting τ_i equal to the largest element among those smaller than or equal to $\min\{\max\{\pi_1, \dots, \pi_i\}, \max\{\rho_1, \dots, \rho_i\}\}$ not yet appeared in the previous positions. For instance, given $\pi = 32657481, \rho = 24378651$ we get $\pi \vee \rho = 34678521$ and $\pi \wedge \rho = 23467581$. In the following proposition we show that the above defined operations actually coincide with the join and meet operations in $S_n(312)$.

Proposition 4.1.4 *For any $\pi, \rho \in S_n(312)$, the permutations $\pi \vee \rho$ and $\pi \wedge \rho$ are respectively the join and the meet of π and ρ in the Bruhat lattice $S_n(312)$.*

Proof. Let $\max(\pi)$ and $\max(\rho)$ be the max-vectors of the noncrossing partitions associated with π and ρ, respectively. The join of the two Dyck paths associated with π and ρ corresponds to the Dyck path determined by the coordinatewise join of $\max(\pi)$ and $\max(\rho)$, say $\max(\pi) \vee \max(\rho)$, which is then the max-vector of the join of π and ρ in $S_n(312)$. There is a unique 312-avoiding permutation associated with $\max(\pi) \vee \max(\rho)$, which can be obtained as follows: the i-th entry of the permutation is the largest element among those smaller than or equal to the i-th component of the max-vector not yet appeared in the permutation. This corresponds precisely to our definition of $\pi \vee \rho$. The argument for the meet is completely analogous, and so the proof is complete.

□

The above results on 312-avoiding permutations give some useful information on the order structure of $S_n(\tau)$, for any $\tau \in S_3$. To this aim, a crucial step is represented by the following general lemma, whose proof can be found in [I].

Lemma 4.1.2 *Let $r, c, i : S_n \longrightarrow S_n$ the reverse, complement and inverse functions on permutations. Then, with respect to the strong Bruhat order, i is an isomorphism, whereas r, c are antiisomorphisms.*

As a consequence, given $S_n(\tau)$, for some $\tau \in S_k$, endowed with the strong Bruhat order, if we consider the reverse of each element, we get $S_n(\rho)$, with $\rho = r(\tau)$, endowed with the dual order. Analogous considerations can be done for the complement and the inverse functions, whence the following proposition holds.

Proposition 4.1.5 *For every $n \in \mathbf{N}$, $S_n(312)$ is order-isomorphic to $S_n(231)$ and order-antiisomorphic to $S_n(132)$ and $S_n(213)$. Therefore all the above posets are distributive lattices. The posets $S_n(123)$ and $S_n(321)$ are not even lattices, since they do not have minimum and maximum, respectively.*

Clearly, thanks to lemma 4.1.2, the posets $S_n(123)$ and $S_n(321)$ are antiisomorphic.

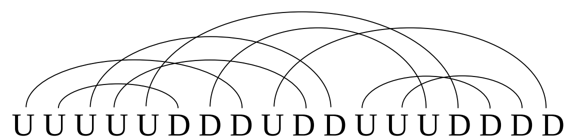
Open problem 1. Describe the poset $S_n(123)$.

Open problem 2. Fixed $k \in \mathbf{N}$, $k > 3$, for which $\tau \in S_k$ is $S_k(\tau)$ a (distributive) lattice under the strong Bruhat order? In case of a positive answer, is it possible to give some alternative combinatorial descriptions of such lattices? We point out that this problem has been solved in [Dr] for the weak Bruhat order.

4.1.5 A possible extension for further work

At the end of our work, we would like to outline the possibility of extending the present study to the class of unrestricted set partitions. More precisely, it would be nice to find an order structure on set partitions coinciding with the Bruhat order when restricted to noncrossing partitions, as well as to determine a suitable class of paths associated with such an order and to explore the connections with pattern avoiding permutations. Concerning this last statement, observe that, if we agree to represent unrestricted partitions analogously to noncrossing ones (i.e., the elements inside each block are in decreasing order, and the blocks are listed in such a way that their maxima are increasing), then there is an obvious bijection between set partitions and 3-12 avoiding permutations (just adapt the argument of Proposition 4.2.2). Unfortunately, the order structure induced by the strong Bruhat order on S_n(3-12) is not a lattice in general (for instance, when $n = 4$, the two permutations 3142 and 2341 do not have a greatest lower bound).

A first step towards this direction will be the object of the last result of this work. Consider the set Ω of Dyck words, that is the set of all the words ω of the two-letter alphabet $\{U, D\}$ satisfying the well-known Dyck condition, i.e. ω contains the same number of D's and U's, and every prefix of ω contains at least as many U's as D's. We call a *matching* of a Dyck word $\omega = \omega(U, D)$ any matching between the U's and the D's of ω. We represent matchings by arc diagrams as in the figure below:



We define a *Bell matching* to be a matching of $\omega \in \Omega$ satisfying the following two conditions:

1. for any set of consecutive D's, the leftmost D is matched with the adjacent U on its left;
2. every other D is matched with a U on its left, in such a way that there are no crossings among the arcs originated from a set of consecutive D's.

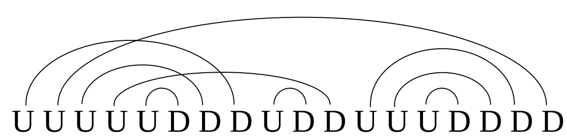


Figure 4.6 A Bell matching of a Dyck word of length 18

The next proposition shows the connection between Bell matchings and set partitions.

Proposition 4.1.6 *There is a bijection between Bell matchings of Dyck words of length* $2n$ *and set partitions of* $[n]$.

Proof. Given a Dyck word ω, enumerate the U's of ω from left to right, then label each D with the number of the U is it matched with. The sets of the labels of each subword of consecutive D's constitute the block of a set partition. It is easy to see that this construction can be reversed to get the desired bijection.

□

Observe that noncrossing partitions correspond to Bell matchings without crossings. Clearly, for every Dyck word of length $2n$, there is precisely one Bell matching without crossings.

The class of Bell matchings of Dyck words of any fixed length can be partitioned as follows. We declare two Bell matchings *equivalent* when they are matchings of the same Dyck word. This equivalence relation can be translated on set partitions, in such a way that each equivalence class contains precisely one noncrossing partition. Our final result is the enumeration of set partitions by counting the elements in each equivalence class. As a consequence, we get an expression of Bell numbers in terms of natural parameters of Dyck paths (height of peaks and valleys) which we believe to be new.

Theorem 4.1.5 *Let $\pi = B_1|\dots|B_k \in NC(n)$. Then the equivalence class $[\pi]$ has cardinality:*

$$|[\pi]| = \prod_{i=1}^{k} \binom{b_i - \sum_{j=1}^{i-1} |B_j| - 1}{|B_i| - 1}, \tag{4.4}$$

where $b_i = \max B_i$, as usual. Equivalently, the bijection between noncrossing partitions and Dyck paths translates formula (4.4) into the following:

$$|[P]| = \prod_{i=1}^{k} \binom{p_i - 1}{v_i}, \tag{4.5}$$

where P is the Dyck path associated with π and p_k and v_k are the heights of the k-th peak and of the k-th valley of P, respectively.

Summing up the cardinalities of all the equivalence classes, we get the following expression for Bell numbers:

$$\mathcal{B}_n = \sum_{P \in \mathcal{D}_n} \prod_{i=1}^{k} \binom{p_i - 1}{v_i}. \tag{4.6}$$

Proof. Given the noncrossing partition π, we can obtain formula (4.4) by considering the Dyck word ω associated with π and then counting the number of Bell matchings definable on such a word. To this aim, consider the first set of consecutive D's in ω. The starting D has to be matched with the adjacent U, so there is only one possible choice for it; all the remaining D's of this first group can be matched with any of the preceding U's. Since the cardinality of the starting factor of U's is b_1 (i.e., the maximum of the block B_1), and the cardinality of the first set of consecutive D's is $|B_1|$, we have $\binom{b_1-1}{|B_1|-1}$ possible choices. Now consider the second factor of consecutive D's in ω. In this case, we have to match all these D's with any of the preceding U's not previously matched, except for the leftmost D, which must be matched with the adjacent U. Since we have to choose $|B_2|-1$ U's out of a set of $b_2-|B_1|-1$ U's, the possible choices are $\binom{b_2-|B_1|-1}{|B_2|-1}$. Iterating this argument we eventually get formula (4.4), as desired.

To obtain formula (4.5) we have to translate parameters on noncrossing partitions into parameters on Dyck paths, as we did in Theorem 4.7.

□

4.2 Order properties of the Motzkin and Schröder families

The main goal of the present section is to find analogous results starting from the distributive lattices of Motzkin and Schröder paths. More precisely, we aim at finding suitable modifications of the above described bijections which allow us to obtain distributive lattice structures on some kind of noncrossing partitions and pattern avoiding permutations having some combinatorial relevance. In the Motzkin case, our results are reported in Section 4.2.1 and are strikingly similar to those of the Catalan family. Our basic tool is a bijection described in [EM2] which codifies Motzkin paths by means of a special kind of Dyck paths. Moreover, our main result is the fact that $S_n(31-2), k-(k-1)(k-2)\dots21$ is a distributive lattice (endowed with the strong Bruhat order) for every $k \geq 2$; to the best of our knowledge, this is a new result of order-theoretic flavor concerning classes of pattern avoiding permutations.

In the Schröder case, things are not so neat, and we need to introduce coloured objects to achieve some satisfactory results (which are described in Section 4.2.2). The last section is devoted to the presentation of some open problems (many of which are also scattered throughout the section), as well as of some possible directions of future research.

At the end of this introduction, we give explanations concerning some notations we are using.

The word "bar" is used to denote both vertical and horizontal bars, so that its meaning depends on the context. When we speak of "bar-removing bijection", we mean the function which removes the *vertical* bars in the standard notation of a partition to obtain a permutation, whereas the terms "barring" and "unbarring" indicate the operation of putting and removing a horizontal bar over an element of a permutation. However, the choice between vertical and horizontal should be clear from the context.

The sequences of Schröder and Narayana numbers will be denoted by $(R_n)_{n\in\mathbf{N}}$ and $(N(n,k))_{n,k\in\mathbf{N}}$, respectively.

The up, horizontal and down steps in Dyck, Motzkin and Schröder paths will be denoted u, h, d, respectively. A Dyck path of length $2n$ is a lattice path consisting of u and d steps, from $(0,0)$ to $(2n,0)$ which never pass below the x-axis. A Motzkin path of length n is a lattice path which uses u, d and h steps, from $(0,0)$ to $(n,0)$, never passing below the x-axis. A Schröder path of length $2n$ is a lattice path starting from $(0,0)$ to $(2n,0)$ consisting of u, d and hh (double horizontal) steps, never going below the x-axis.

The symmetric group on n elements will be denoted by S_n, whereas the set of coloured permutations on n elements will be denoted by $\overline{S}_n$.

4.2.1 Motzkin paths

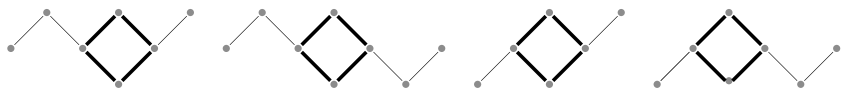
We start by recalling a bijection introduced by Elizalde and Mansour [EM] between the set $\mathcal{M}_n$ of Motzkin paths of length n and the set $\mathcal{D}_n^{(3)}$ of Dyck paths of length $2n$ without three consecutive down steps. Every Dyck path $P \in \mathcal{D}_n^{(3)}$ can be uniquely decomposed into factors of the following three types: u, ud, udd. Define a Motzkin path $f(P)$ by translating the above factors according to the following:

$$\begin{aligned} u &\rightarrow u \\ ud &\rightarrow h \\ udd &\rightarrow d \end{aligned}$$

$f(P)$ has length n and it is possible to show that the function f is a bijection. Our next proposition shows that f has some more structural properties.

Proposition 4.2.1 *The function $f : \mathcal{D}_n^{(3)} \longrightarrow \mathcal{M}_n$ is an order-isomorphism.*

Proof. Let $P, Q \in \mathcal{D}_n^{(3)}$ such that $P \preceq Q$. This means that Q is obtained from P by changing a valley into a peak. Call *box* the two steps on which P and Q differ. However, we notice that, unlike it happens for the whole $\mathcal{D}_n$, in some cases performing the above operation on paths belonging to $\mathcal{D}_n^{(3)}$ does not produce a path of the same kind: this occurs precisely when a valley is followed by two or more down steps. When we apply f to P and Q, several different things can happen, according to the type of the steps next to the box. Since the down step of a valley cannot be preceded by two or more down steps, there are only two possibilities for P and Q, namely the box is preceded either by u or by ud. Analogously, the down step of a peak cannot be followed by two or more down steps, whence also in this case we have two different cases, i.e. the box is followed either by u or by du. Therefore we have a total of four cases, depicted in the figure below:



Now apply f to each of the above, to obtain respectively the following four cases on the corresponding Motzkin paths:

As it is clear, each situation yields two Motzkin paths $f(P), f(Q)$ such that $f(P) \preceq f(Q)$, as desired.

Conversely, an analogous argument shows that, if P, Q are arbitrary Motzkin paths for which $P \preceq Q$, then $f^{-1}(P) \preceq f^{-1}(Q)$, so the proof is complete.

□

The bijection between $\mathcal{D}_n$ and $NC(n)$ recalled in the introduction can be restricted to $\mathcal{D}_n^{(3)}$; the corresponding subset of $NC(n)$ is easily seen to consist of noncrossing partitions whose blocks have cardinality at most 2. Call such partitions *Motzkin noncrossing partitions.* Thanks to the last proposition we can establish the following result.

Theorem 4.2.1 *The set $MNC(n)$ of Motzkin noncrossing n-partitions can be endowed with a distributive lattice structure, which is isomorphic to the lattice of Motzkin paths of length n. More precisely, given $\pi, \rho \in MCN(n)$, we have that $\pi \preceq \rho$ if ρ is obtained from π by moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ if $\beta = \min \tilde{B}$. In this case, either:*

1. *keep β inside $\tilde{B}$, if $|\tilde{B}| = 1$, or*

2. add a new block $\overline{B} = \{\beta\}$*, if* $|\tilde{B}| = 2$*.*

Proof. The first part of the theorem is an easy consequence of proposition 4.2.1. As far as the covering relation is concerned, the above result for Motzkin noncrossing partitions immediately derives from the analogous one given in [BBFP] for general noncrossing partitions. The only thing to take into account is that a Motzkin noncrossing partitions has blocks of cardinality at most 2, and so, if $|\tilde{B}| = 2$ and $\beta = \max \tilde{B}$, the above mentioned rule cannot be applied since the resulting partition would not belong to $MNC(n)$.

□

Example. Given the partition $2|31|65|74|8 \in MNC(n)$, there are two partitions covering it, which are $2|3|4|65|71|8$ (1 has been moved into a block with two elements) and $2|31|65|7|84$ (4 has been moved into a block with one element). Note that the partitions obtained by moving 2 or 5 are not listed above, since the elements 3 and 7 are not the minima of their blocks.

Remark. Another consequence of proposition 4.2.1 is that the rank of a partition of $MNC(n)$ corresponds to the area of the associated Motzkin path, this meaning that two partitions of $MNC(n)$ have the same rank if and only if the associated Motzkin paths have the same area. Also in this case, a formula expressing the area using parameters on partitions (such as cardinality of a block and maximum of a block) can be found as in [BBFP].

Similarly to [BBFP], it is possible to transfer the distributive lattice structure of Motzkin noncrossing partitions to a suitable subset of pattern avoiding permutations via a bar-removing bijection. In [C] it is shown (bijectively) that $S_n(3-21, 31-2)$ is counted by Motzkin numbers. Here we give an essentially equivalent bijection between $MNC(n)$ (and so Motzkin paths) and $S_n(3-21, 31-2)$.

Proposition 4.2.2 *Removing the vertical bars in Motzkin noncrossing partitions defines a bijection between $MNC(n)$ and the set $S_n(3-21, 31-2)$ of pattern avoiding permutations of $[n]$, for any $n \in \mathbf{N}$.*

Proof. Let π be a permutation of $S_n(3-21, 31-2)$. It is straightforward to see that the associated partition is a noncrossing partition, since π avoids the pattern $31-2$ ([BBFP]). If π contains a block with three or more elements, then the associated permutation would show the forbidden pattern $3-21$, against the hypothesis. So π is a Motzkin noncrossing partition. On the other hand, if $\pi \in MNC(n)$, then the associated permutation avoids the pattern $31-2$. Moreover, if π contains a pattern $3-21$ in the entries π_i, π_k, π_{k+1}, then necessarily $\pi_{k-1} < \pi_k$, otherwise π would have a block with three elements. So the entries π_i, π_{k-1} and π_k are a pattern $3-12$ which induces the presence in π of the forbidden pattern $31-2$ (see [BFP]). We conclude that $\pi \in S_n(3-21, 31-2)$.

□

To prove that the above bar-removing bijection between $MNC(n)$ and $S_n(3-21, 31-2)$ is also an order-isomorphism, we just notice that such a bijection is obtained by simply restricting the bar-removing isomorphism between $NC(n)$ and $S_n(312)$ considered in [BBFP]. Therefore the following theorem holds.

Theorem 4.2.2 *Let $(S_n(3-21, 31-2); \leq)$ be the poset obtained by transferring the distributive lattice structure defined in 4.2.1 along the bar-removing bijection. This is precisely the subposet induced on $S_n(3-21, 31-2)$ by the strong Bruhat order of the symmetric group S_n. Therefore $S_n(3-21, 31-2)$ is a distributive sublattice of S_n endowed with the strong Bruhat order.*

An immediate consequence of the above theorem is stated in the following, remarkable corollary.

Corollary 4.2.1 *For any $n \in \mathbf{N}$, the Motzkin lattice $\mathcal{M}_n$ is isomorphic to the lattice $S_n(3-21, 31-2)$ with the strong Bruhat order.*

We conclude this section by generalizing the bijection of Elizalde and Mansour between $\mathcal{D}_n^{(3)}$ and $\mathcal{M}_n$. Denote by $\mathcal{D}_n^{(k)}$ the set of Dyck paths of length $2n$ having at most $k-1$ consecutive down steps and by $\mathcal{C}_n^{[-k+2,1]}$ the set of paths of length n starting from the origin, ending on the x-axis, never falling below the x-axis and using steps of the kind $(1, j)$, for $j \in \{-k+2, -k+1, \ldots, -1, 0, 1\}$ (this notation is borrowed from [FP2]). Each path in $\mathcal{D}_n^{(k)}$ can be uniquely factorized using factors of type ud^j, for $0 \leq j \leq k-1$. Therefore we can define a bijection analogous to f by mapping the factor ud^{j+1} into the step $(1, -j)$, thus obtaining a path in $\mathcal{C}_n^{[-k+2,1]}$. Call f_k such a bijection (with this notation, clearly $f = f_3$). Using an argument similar to proposition 4.2.1, it is possible to show that f_k is an order-isomorphism, for any $k \geq 2$. Moreover, from a general result proved in [FP2], each set of paths $\mathcal{C}_n^{[-k+2,1]}$ is a distributive lattice with the usual order. As a consequence, our previous results on the order structure of paths, partitions and permutations counted by Motzkin numbers can be extended as follows:

Proposition 4.2.3 *For any $k \geq 2$, the following distributive lattice structures are isomorphic:*

1. *$\mathcal{C}_n^{[-k+2,1]}$ with the usual order on paths;*

2. *the set $kNC(n)$ of noncrossing partitions of an n-set having blocks of cardinality at most $k-1$, endowed with the order inherited by the Bruhat order of $NC(n)$;*

3. *the set of generalized pattern avoiding permutations $S_n(31-2, k-(k-1)(k-2)\cdots 21)$ endowed with the strong Bruhat order.*

When k tends to infinity, we get a bijection f_∞ between Dyck paths of length $2n$ and paths of length n using the unique positive step $(1, 1)$ and any kind of negative step $(1, -j)$. This latter class of paths will be called here the class of *Lukasiewicz* paths. Observe that Lukasiewicz paths are usually defined dually (in [BF] they correspond to our paths read from right to left), anyway both enumerative results and order properties are not affected by this slight change of notation. The above proposition translates into the fact that the distributive lattices of Lukasiewicz paths are isomorphic to those of Dyck paths, as well as to the Bruhat noncrossing partition lattices and 312-avoiding permutations with the strong Bruhat order.

From an enumerative point of view, we observe that for $k = 2$ we get the sequence $1, 1, 1, \ldots$, for $k = 3$ we get the Motzkin numbers and for $k = \infty$ we get the Catalan numbers. Therefore the sequences obtained for a generic k interpolate between the Motzkin and the Catalan numbers. A strikingly similar result has been found in [BDPP2], where the authors use classes of pattern avoiding permutations different from ours: it would be interesting to relate the two approaches.

4.2.2 Schröder paths

In this section we try to find analogous results starting from the lattices of Schröder paths.

A first attempt in this direction consists of reading Schröder paths as special Motzkin paths, namely a Schröder path can be regarded as a Motzkin path in which any set of consecutive horizontal steps has even cardinality. From this point of view, we can consider a suitable restriction of the bijection of proposition 4.2.1. As a consequence of this approach, we obtain that Schröder lattices are isomorphic to the lattices of Motzkin noncrossing partitions where any bunch of singletons made of consecutive integers has even cardinality. Unfortunately, we have not been able to determine the set of pattern avoiding permutations associated with the above subset of Motzkin noncrossing partitions via the bar-removing bijection.

Open problem 3. Find a set of patterns T such that $S_n(T)$ corresponds to the set of Schröder paths of length n via a suitable restriction of the bijection between Dyck paths and 312-avoiding permutations recalled in the introduction.

A totally different approach consists of interpreting Schröder paths as Dyck paths with bicoloured peaks. Denote by $\overline{\mathcal{D}}_n$ the set of Dyck paths of length $2n$ whose peaks can be coloured either white or black. There is an obvious bijection between $\overline{\mathcal{D}}_n$ and the set $\mathcal{S}_n$ of Schröder paths of length $2n$ (just map white peaks into simple peaks, black peaks into a pair of consecutive horizontal steps, and leave the remaining steps unchanged; from this bijection, which has been considered in [Su], the identity $R_n = \sum_{k=1}^{n} 2^k N(n,k)$ immediately follows). Thanks to this simple observation, it is not difficult to find a suitable set of coloured noncrossing partitions in bijection with Schröder paths.

Proposition 4.2.4 *Denote by $\overline{NC}(n)$ the set of noncrossing partitions of an n-set such that the maximum of the blocks can be either coloured white or black. Then there is a bijection between $\mathcal{S}_n$ and $\overline{NC}(n)$.*

Proof. Given a Schröder path, consider the associated bicoloured Dyck paths and take the noncrossing partition determined by the classical bijection, taking care of colouring each element of the partition with the same colour of the corresponding down step.

□

An example illustrating the bijections connecting $\mathcal{S}_n$, $\overline{\mathcal{D}}_n$ and $\overline{NC}(n)$ for $n = 6$ is given in figure 1.

The elements of $\overline{NC}(n)$ will be called *Schröder noncrossing partitions.* From now on, in a Schröder noncrossing partition we will denote black elements using a horizontal bar, and we will simply call them *coloured* elements.

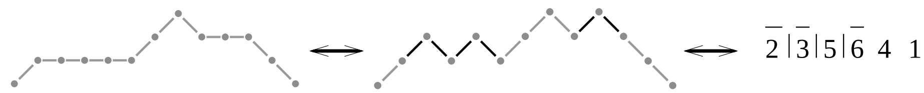


Figure 4.7 The bijections connecting $\mathcal{S}_n$, $\overline{\mathcal{D}}_n$ and $\overline{NC}(n)$ for $n = 6$

Similarly to Dyck paths, Schröder paths can be endowed with a natural partial order structure, and the obtained poset is again a distributive lattice [FP2]. Here we only recall the covering relation: if U is a Schröder path, then a path V covering it ($U \preceq V$) is obtained either by:

- changing a pair du in U into a pair hh in V, or
- changing a pair hh in U into a pair ud in V. Note that, in this second case, the replacement is possible only if the hh in U is followed by an even number of h steps, otherwise the path V would not be a Schröder path.

The natural order on Schröder paths of length $2n$ can be transferred to $\overline{NC}(n)$ by means of the bijection of proposition 4.2.4. We have the following theorem:

Theorem 4.2.3 *(Characterization of coverings) Given two coloured non-crossing partitions $\pi, \rho \in \overline{NC}(n)$, we have $\pi \preceq \rho$ if and only if ρ is obtained from π by either*

1. *unbarring a coloured element of π, or*
2. *moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ only when β is not coloured; moreover:*
   - *(a) if $\beta = \max \tilde{B}$, then keep β inside $\tilde{B}$ and bar it;*
   - *(b) if $\beta \neq \max \tilde{B}$, then add the coloured block $\overline{B} = \{\overline{\beta}\}$.*

Proof (sketch). We can proceed as we did in theorem 4.1 of [BBFP] for the covering relation on $NC(n)$, so we omit a detailed proof. However, it is worth noticing that the bijection between $\mathcal{S}_n$ and $\overline{\mathcal{D}}_n$ implies that if $P, Q \in \overline{\mathcal{D}}_n$ are such that $P \preceq Q$, then Q is obtained from P by either changing a black peak into a white peak or replacing a valley with a black peak (observe that this last operation on valleys can be only performed when the steps are both white).

$\square$

Example. Given the partition $\bar{5}43|62|871|\bar{9} \in \overline{NC}(n)$, there are precisely four partitions covering it, which are $543|62|871|\bar{9}$ ($\bar{5}$ has been unbarred), $\bar{5}4|\bar{6}32|871|\bar{9}$ (3 has been moved and 6 was the maximum of its block), $\bar{5}43|6|\bar{7}|821|\bar{9}$ (2 has been moved and 7 was not the maximum of its block) and $\bar{5}43|62|871|9$ ($\bar{9}$ has been unbarred). Note that the partition obtained by moving 1 into the block containing $\bar{9}$ (i. e. the maximum of its block plus 1) is not listed above, since $\bar{9}$ is coloured.

The area $A(P)$ of a Schröder path P can be derived from the Dyck path P' obtained by replacing each double horizontal step with a coloured peak. If C is the number of coloured peaks of P', then it is easily seen that $A(P) = A(P') - C$. Now, the rank of the associated Schröder noncrossing partition π can be expressed by recalling the formula in [BBFP] for the rank of a noncrossing partition. Denoting by $\pi' \in NC$ the (noncoloured) noncrossing partition associated with π, we have

$$A(\pi') = \sum_{i=1}^{k} \left(|B_i| \left(2b_i - 2\sum_{j=1}^{i-1} |B_j| - |B_i| \right) \right)$$

whence the rank of π is given by:

$$r(\pi) = A(\pi') - c(\pi) \quad ,$$

where $c(\pi)$ is the number of coloured elements of π.

Following the lines of [BBFP], we now look for a suitable set of coloured pattern avoiding permutations in bijection with both Schröder paths and

Schröder noncrossing partitions. The study of the enumerative properties of coloured pattern avoiding permutations has been pursued by several authors, see for example [M]. The next result has been independently proved by Egge [E] using algebraic arguments; here we propose a bijective proof, as well as a presumably new order structure connecting a certain class of permutations with Schröder paths and Schröder noncrossing partitions.

Theorem 4.2.4 *Removing the vertical bars in Schröder noncrossing partitions defines a bijection between* $\overline{NC}(n)$ *and the set* $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$*, for any* $n \in \mathbf{N}$.

Proof. Let π be a partition of $\overline{NC}(n)$. We show that π avoids the four patterns $312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12$.

If π' denotes the associated permutation via the bar-removing bijection, then it is known [BBFP] that π' is a 312-avoiding permutation, since π is a noncrossing partition (just recall the standard representation of partitions given in the introduction).

Suppose that π' contains $\bar{2}\bar{1}$. Since in π only the maxima of the blocks can be coloured, it means that π contains two maxima in decreasing order, which is not possible due to our standard notation.

If π' contains $2\bar{1}$ in its elements a and $\bar{b}$, with $a > \bar{b}$, then, regarded as elements of π, they belong to two different blocks and $\bar{b}$ is the maximum of its block. Then, considering $\bar{b}$ and the maximum of the block containing a, two maxima in decreasing order would appear in π, against the hypothesis.

Let us suppose that π' contains a $\bar{3}12$ pattern in the elements $\bar{a}$, b and c, with $\bar{a} > c > b$. Then, in π, b and c lie in two different blocks. Suppose that $\bar{a}$ is the maximum of the block containing b. Let d be the maximum of the block containing c. Clearly $d > a$, since maxima are in increasing order. The elements $\bar{a}, b, c, d$ constitute a crossing being $\bar{a}$ in the same block of b, d in the same block of c and $b < c < a < d$. This is not possible since $\pi \in \overline{NC}(n)$. If $\bar{a}$ is not the maximum of the block of b, the same argument of the previous point can be repeated considering the maximum g of the block containing b. So π' is also a $\bar{3}12$-pattern avoiding permutation, whence $\pi' \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$.

Vice versa, given $\pi' \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$, consider the partition π obtained by inserting a vertical bar before each left-to-right maximum other than the first one. In this way, the maxima of the blocks of π are precisely the left-to-right maxima of π'. Moreover, the fact that π' avoids the two patterns $\bar{2}\bar{1}, 2\bar{1}$ implies that the only elements of π which can be coloured are the maxima of its blocks. Finally, the avoidance of the two patterns $312, \bar{3}12$ forces the partition π to be both in standard notation and noncrossing.

□

Remark. The above set of coloured pattern avoiding permutations clearly coincides with $\Theta_n(\bar{2}\bar{1}, 2\bar{1})$, where Θ_n is the set of coloured permutations of length n avoiding any coloured version of the pattern 312 (and so $|\Theta_n| = 2^n C_n$).

Using the above bar-removing bijection we can now transfer the order structure of Schröder paths to the set $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$. What we obtain is clearly a distributive lattice; its covering relation is described in the next proposition, whose proof is omitted.

Proposition 4.2.5 *Given $\pi, \rho \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$, it is $\pi \prec \rho$ if and only if ρ is obtained from π by either:*

1. *unbarring an element of π, or*
2. *interchanging the element a immediately preceding a left-to-right maximum of π with $\beta + 1$, where β is the left-to-right maximum before a, and colouring $\beta + 1$; this last operation can be performed exclusively when a and $\beta + 1$ are both unbarred.*

Example. The reader can reconsider the example presented at the end of theorem 4.2.3: just delete the vertical bars and read the covering rules according to the last proposition.

Remark. We recall that it is possible to define a notion of Bruhat order on coloured permutations, as it is reported, for instance, in [BB]. Unfortunately, the restriction of this Bruhat order to $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ does not match our posets.

Open problem 4. Concerning the above remark, the Bruhat order on $\overline{S}_n$ is defined in [BB] as the Bruhat order on the set of permutations with ground set $\{1, \dots n, \bar{1}, \dots \bar{n}\}$, where the elements are linearly ordered as they are listed above (i.e., $1 < \cdots < n < \bar{1} < \cdots < \bar{n}$). Is it possible to find a suitable linear order on $\{1, \dots n, \bar{1}, \dots \bar{n}\}$ such that the resulting Bruhat order on $\overline{S}_n$ coincides with our partial order?

Let $\pi \in \overline{S}$; we denote by $inv(\pi)$ the set of the inversions of π and $nb(\pi)$ the number of the unbarred entries of π. Then the following proposition holds:

Proposition 4.2.6 *The rank of an element* $\pi \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ *is given by*

$$r(\pi) = 2|inv(\pi)| + nb(\pi)\ . \tag{4.7}$$

Proof. We proceed by induction.

If $r(\pi) = 0$, then $\pi = \bar{1}\bar{2} \dots \bar{n}$ and $inv(\pi) = \emptyset$, $nb(\pi) = 0$, whence formula (4.7) is true.

Suppose that $r(\pi) = 2|inv(\pi)| + nb(\pi)$ for $\pi \in \overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ such that $r(\pi) = s$. Let ρ be a permutation of $\overline{S}_n(2\bar{1}, \bar{2}\bar{1}, 312, \bar{3}12)$ such that $\pi \preceq \rho$, then $r(\rho) = s + 1$. We have to show that $r(\rho) = 2|inv(\rho)| + nb(\rho)$. There are two possibilities for ρ: ρ is obtained from π either by unbarring an element or by interchanging the elements of a pattern 12 of π obeying condition 2 of proposition 4.2.5 to obtain a pattern $\bar{2}1$ in ρ (in this way ρ has precisely one more inversion than π). In the first case $inv(\rho) = inv(\pi)$ and $nb(\rho) = nb(\pi) + 1$. Then,

$$r(\rho) = r(\pi) + 1 = 2|inv(\pi)| + nb(\pi) + 1 = 2|inv(\rho)| + nb(\rho) \ .$$

In the second case $|inv(\rho)| = |inv(\pi)| + 1$ and $nb(\rho) = nb(\pi) - 1$. Then,

$$r(\rho) = 2|inv(\pi)|+nb(\pi)+1 = 2(|inv(\rho)|-1)+nb(\rho)+1+1 = 2|inv(\rho)|+nb(\rho) \ .$$

In both cases, formula (4.7) holds.

□

4.2.3 Hints for further work

In this last section we propose some ideas to get a better insight into the properties of the above considered order structures.

Given a Dyck path P of length $2n$, it is very natural to consider the Dyck path $m(P)$ obtained by reading P from right to left. So, for example, if $P = uuuuudduddudd d$, then $m(P) = uuuduuduudd ddd$. The function m maps $\mathcal{D}_n$ into itself, and it is clearly an involution which preserves the area, therefore it is a rank-preserving involution. More precisely, m is an order-isomorphism of $\mathcal{D}_n$. Therefore, if we transfer m to $NC(n)$ and $S_n(312)$, we obtain an order-isomorphism (still to be denoted m) of both the Bruhat noncrossing partition lattice of order n and the set of 312-avoiding permutations of length n with the strong Bruhat order. The next proposition allows to determine $m(\pi)$ for any $\pi \in NC(n)$. The translation of this result on $S_n(312)$ is straightforward.

Proposition 4.2.7 *Let* $\pi = B_1|B_2|\cdots|B_k \in NC(n)$.
Then $m(\pi) = C_1|C_2|\cdots|C_k \in NC(n)$ *where* $|C_i| = \max B_{k-i+1} - \max B_{k-i}$ *and* $\max C_i = \sum_{j=k-i+1}^{k} |B_j|$.

Proof. First of all we observe that a noncrossing partition is uniquely determined by the cardinalities and the maxima of its blocks.

Let P be the Dyck path associated with π. By definition, the partition $m(\pi)$ is obtained by numbering the down steps of P in decreasing order, then labelling each of its up steps with the number of the down step it is matched with and taking as blocks the sets of labels of consecutive sequences of up steps. Now suppose that $m(\pi) = C_1|C_2|\cdots|C_k$ is written in standard notation, as usual. Since the difference between the maxima of two consecutive blocks B and B' of π represents the number of consecutive up steps of P between the two sequences of down steps corresponding to B and B', it is clear that $|C_i| = \max B_{k-i+1} - \max B_{k-i}$. Moreover, the maximum c of a block of $m(\pi)$ coincides with the number of down steps of P following the up step corresponding to c, and so $\max C_i = \sum_{j=k-i+1}^{k} |B_j|$.

□

It is clear that an analogous involution can be defined also for Motzkin and Schröder paths. As far as Motzkin paths are concerned, there are two possible approaches. First, given a Motzkin path $P \in \mathcal{M}_n$, one can read it from right to left, so obtaining another Motzkin path of $\mathcal{M}_n$. On the other hand, one can restrict m to the set $\mathcal{D}_n^{(3)}$ of Dyck paths of length $2n$ having at most two consecutive down steps. In this way, the image of m is the set ${}^{(3)}\mathcal{D}_n$ of Dyck paths without three consecutive up steps. Anyway, both in the Motzkin and Schröder case, it seems not too difficult to find results on partitions and permutations analogous to the last proposition.

A much more difficult task consists of interpreting the bar-removing bijection in an alternative way. More precisely, given a noncrossing partition π written in standard notation, we associate with it the permutation obtained by reading each block of π as a cycle. For instance, the partition $543|62|871|9$ is mapped into the permutation $(543)(62)(871)(9)$. It is evident that the permutations obtained in this way have a special cycle structure [MacC]; it would be interesting to see if such a structure can be expressed in terms of (possibly generalized) pattern avoidance. Moreover, transferring to this set of permutations the order structure of Dyck paths leads to a new partial order on permutations, whose properties are probably worth being investigated.

We point out that the above map from noncrossing partitions to permutations written in cycle form has already been considered in [MacC], where the author describes the partial order obtained in S_n by transferring the *refinement order* of NC_n.

Chapter 5

About the generation of combinatorial objects

This chapter contains some hints which could be developed in order to investigate on the generation of pattern avoiding permutations. More precisely, starting from a succession rule for the Catalan numbers (which enumerate the permutations avoiding a pattern of length three), we define a procedure encoding and listing the objects enumerated by these numbers such that two consecutive codes of the list differ only for one digit (Section 5.1). Gray code we obtain can be generalized to all the succession rules with the *stability property*: each label (k) has in its production two labels c_1 and c_2, always in the same position, regardless of k. Because of this link, we define *Gray structures* the sets of those combinatorial objects whose construction can be encoded by a succession rule with the stability property. This property is a characteristic that can be found among various succession rules, as the finite, factorial or transcendental ones. We also indicate an algorithm which is a very slight modification of the Walsh's one, working in a $O(1)$ worst-case time per word for generating Gray codes.

Subsequently (Section 5.2), we propose a procedure to generate all Dyck paths of length n. The CAT generation algorithm we deduce formalizes a method for the exhaustive generation of these paths which can be described by two equivalent strategies of construction, based on the ECO method. A very slight modification of our procedure allows to extend it to the generation of other paths (Grand Dyck and Motzkin paths). We think that a similar approach can be used also for the permutations avoiding one pattern or other classes of permutations. The first idea in this sense could be to use the bijection between Dyck paths and 312-avoiding permutations (se Section 4 or [Si]).

5.1 A general exhaustive generation algorithm for Gray structures

5.1.1 Introduction

The matter of encoding and listing the objects of a particular class is common to several scientific topics, ranging from computer science and hardware or software testing to chemistry, biology and biochemistry. Often, it is very useful to have a procedure for listing or generating the objects in a particular order. A very special kind of list is the so called Gray code, where two successive objects are encoded in such a way that their codes differ as little as possible (see below for more details and [Wa]). There are many applications of the theory of Gray codes for several combinatorial objects, involving permutations [J], binary strings, Motzkin and Schröder words [V2], derangements [BV], involutions [Wa1], P-sequences [V1]. They are also used in other technological subjects as circuit testing, signal encoding [Lud], data compression and other (we refer to [BBGP] for an exhaustive bibliography on the general matter).

The generation of a Gray code is often strictly connected with the nature of the objects which we are dealing with. So, it seems to have some importance the definition of a Gray code for the objects of the classes with some common characteristic, as the classes enumerated by the same sequence. From the idea of [BBGP], which we briefly recall in the sequel, in this work we develop a procedure for listing the objects of those structures whose exhaustive generation can be encoded by particular succession rules (see below), say succession rules satisfying the *stability property* (see Section 5.1.5). In order to point out the relation between such structures and the possibility to list their objects in a Gray code, we define them *Gray structures*.

Our discussion moves from the well known succession rule Ω_C,

$$\Omega_C : \begin{cases} (2) \\ (k) \rightsquigarrow (2)(3)\ldots(k)(k+1), \qquad k \geq 2 \end{cases}$$

defining the sequence of Catalan numbers and whose first levels of the related generating tree are shown in Figure 5.1. Each object x with size n corresponds to a node at level $n-1$ (being the root of the tree at level 0, corresponding to the object of size 1) and can be described by a word $w = w_1 w_2 \ldots w_n$ encoding the path from the root to the node corresponding to x: each w_i is the label of a node of the path and is generated by Ω_C. In [BBGP] the authors give a method to exhaustively generate all the objects (words) of a given size n which substantially coincide with the reading from left to right in the $(n-1)$-th level of the tree. So, the words at level 3 are generated in the following order (see figure 5.1):

$$2222, 2223, 2232, 2233, 2234, 2322, 2323, 2332, 2333, 2334, 2342, 2343, 2344, 2345.$$

In the above list it is possible that two consecutive words differ more than one digits: for instance, 2223 and 2232 differing in two digits or 2234 and 2322 with three different digits. Our aim is to generate all the words of length n (naturally without repetitions) in such a way that *two consecutive words differ only for one digit*. Such a property is strictly related to the concept of "Gray code", which definition we relate can be found in [Wa]. Substantially, it can be summed up in the following: *a Gray code is an infinite set of word-lists with unbounded word-length such that the Hamming* *distance between any two adjacent words is bounded independently of the word-length* (the Hamming distance is the number of positions in which two words differ). For a complete discussion on Gray codes we refer the reader to the paper of T. Walsh [Wa].

In Section 5.1.2 an informal description of the used strategy for our purpose is presented, referring to objects whose construction can be described by Ω_C. Then, in Section 5.1.3, a rigorous definition of the list (Definition 1) and a proof that it is a Gray code (Theorem 5.1.1) are given. Section 5.1.4 presents the application and the analysis to the particular case of Dyck paths, enumerated by Catalan numbers. Finally, Section 5.1.5 generalizes the construction of the Gray code to those objects whose generation can be described by succession rules with the *stability property*. In that section, we also present some examples of Gray structures.

5.1.2 The procedure

The strategy used in [BBGP] for listing the objects of size n corresponds to a visit of all the nodes at level $n-1$ in the generating tree from left to right. So, after the visit of a subtree T_i is completed, the path from the root to the leftmost node of the successive subtree T_{i+1} has at least two different nodes with respect to those ones of the last path of the preceding subtree T_i. This is due to the fact that the labels of the sons of a node are visited in the same order they have in the production of the succession rule Ω_C, where the list of the successors of a label (k) is $< 2, 3, \ldots, k, k+1 >$.

For our purpose we must check that when a subtree has been completely visited and if v is the last path generated in such a visit, then the successive path w has only one different digit with respect to the digits of v. We now illustrate the procedure we are going to use referring to Figure 5.1, where the words of length 4 are generated.

The first object of the list is the word 2222, corresponding to the path

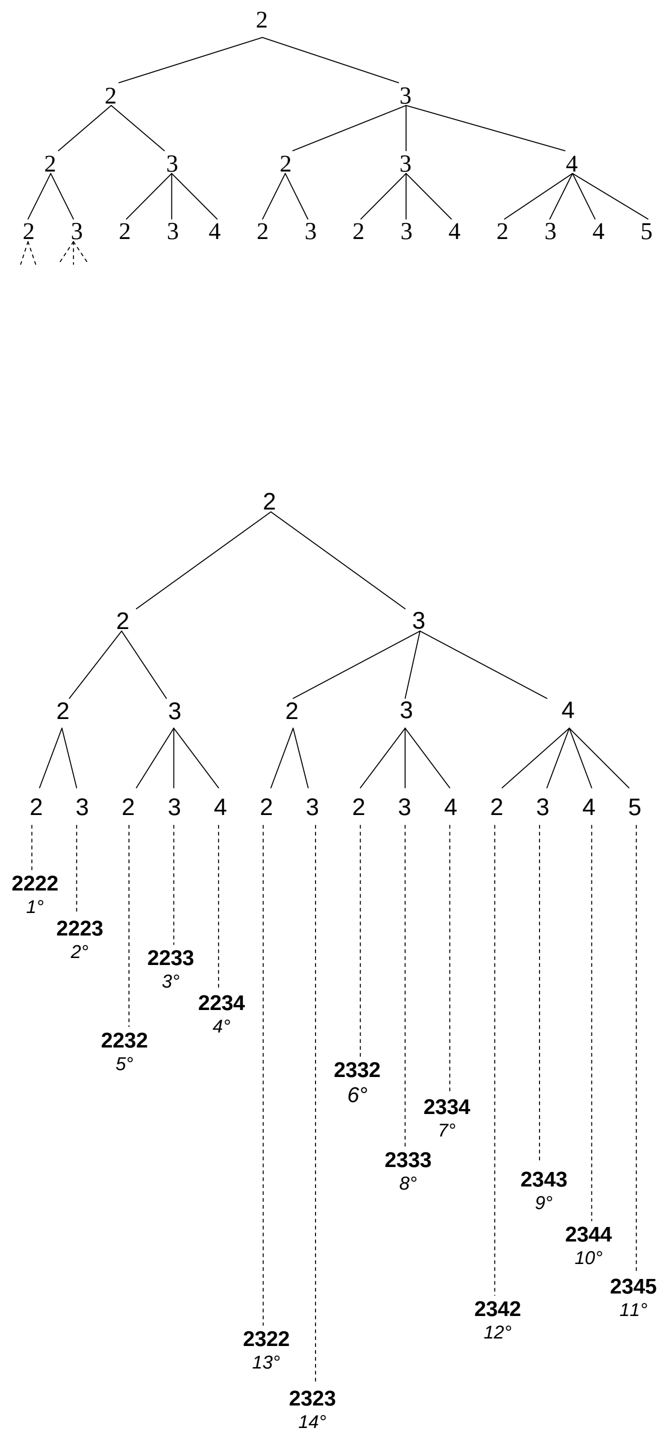


Figure 5.1 First levels of the generating tree for Catalan numbers (upper figure); generation of the words of length 4 (lower figure).

from the root to the leftmost node at level 3 in the generating tree. Then, in order to complete the visit of the current subtree, the second word is 2223. At this point, the next path in the list will have a different digit with respect to the digits of 2223, which is not the last one: in order to respect the above definition of Gray code, the third word in the list could be 2323 or 2233. The choice is determined by the leading idea that a successive path $w = aw_2 \dots w_n$ must have as much as possible the same edges of the preceding path $v = av_2 \dots v_n$ in the list and if v_j and w_j are the first nodes necessarily different in v and w, then all the nodes v_r and w_r must have the same labels for $r = j+1, \dots, n-1, n$, in order to respect the Gray code definition. So, the third word is 2233. The fourth and the fifth one are 2234 and 2232, respectively. From the generation of these last two words we can deduce that only the last digit is changed when a same subtree is visited and that the order for changing the last digit is *shifted* with respect to the classical one in a cyclic way in order to complete the set of the sons of the second-last digit: for the sake of clearness in this case the shifted list of the successors of the second-last digit 3 is $< 3, 4, 2 >$, while the classical one would be $< 2, 3, 4 >$. This fact can be generalized. Let e be the first path of a new subtree and let i and k be the the last and the second-last digit of e, respectively ($i \neq 2$, see below). Then the right order for changing the last digit is $< i, i+1, \dots, k, k+1, 2, 3, \dots, i-1 >$.

The sixth path which is now generated is $f = 2332$, according to the above leading idea. Note that the second digit is changed with respect of the second digit of the fifth word and that the third and the fourth digits in f are the same you find in 2232. The word f is the first path of a new subtree and then only the last digit has to be changed, till the whole set of the sons of the second-last digit 3 is completed. Since the last digit of f is 2, one could think that in this case the *shifted production* of the digit 3 coincides with the classical production $< 2, 3, 4 >$, obtaining that the 6-th, 7-th and 8-th words are 2332, 2333, 2334, respectively. But so doing the procedure fails when it is used to list the words of length 6, as the reader can easily check when he arrives at the generation of the word 234565. The reason of the failure will be clear in the next section, where the rigorous formalization of our procedure is presented. The right way for changing the last digit of f is to consider the list $< 2, 4, 3 >$ of the sons of the digit 3, then obtaining the 6-th, 7-th and 8-th words as follows: 2332, 2334 and 2333, respectively. This fact suggest us that if f is the first word of a new subtree, if its last digit is 2 and if k is its second-last digit, then the right order for changing the last digit is $< 2, k+1, k, k-1, k-2, \ldots, 4, 3 >$. The remaining objects can be now easily obtained, as in Figure 5.1.

We now summarize the definition of the *shifted production* which is used to change the last digit in the words. Let $v = v_1 v_2 \ldots v_n$ be the first path of a new subtree. Let k and i be the second-last and the last digit of v, respectively, then the list $s(k, i)$ of the sons of k such that the first son is i, is:

$$\begin{cases} s(k,2) =< 2, k+1, k, k-1, \ldots 4, 3 > \\ s(k,i) =< i, i+1, \ldots, k-1, k, k+1, 2, 3, \ldots, i-1 > \end{cases} .$$

5.1.3 A Gray code for Catalan structures

First we define the lists for the objects whose generating tree can be described by the succession rule for the Catalan numbers we presented in the previous section, then we will prove (Theorem 5.1.1) that these lists form a Gray code, in the sense of the definition in Section 5.1.1. The following notation is used:

- $\mathcal{L}_k$ = list of the codes of the objects of length k;
- l_i^k = i-th element of $\mathcal{L}_k$;
- $|\mathcal{L}_k|$= cardinality of $\mathcal{L}_k$;

- if x is a sequence of digits, then $\overrightarrow{x}$ is the rightmost digit of x;
- Θ is the concatenation of lists;
- if L is a list, then:
  - $first(L)$ denotes the first element of the list L;
  - $last(L)$ denotes the last element of the list L;
  - $x \circ L$ is the list obtained by pasting x with each element of L.

Our definition is a recursive definition and it is based on a generation of sublists with increasing length:

Definition 1 *The list $\mathcal{L}_n$ of all the elements of length n is*

$$\begin{cases} \mathcal{L}_1 = & < 2 > \\ \mathcal{L}_n = & \Theta_{i=1}^{M} L_n^i \qquad \text{if} \quad n > 1 \end{cases}$$

where $M = |\mathcal{L}_{n-1}|$ and L_n^i is defined by

$$\begin{cases} L_n^1 = & l_1^{n-1} \circ s(2,2) \\ L_n^i = & l_i^{n-1} \circ s(\overrightarrow{l_i^{n-1}}, \overrightarrow{last(L_n^{i-1})}) \quad \text{if} \quad i > 1 \end{cases} .$$

The list L_n^1 is obtained by linking together the first element of the list of the objects of size $n-1$ (i.e. l_1^{n-1}) with the elements of the list $s(2,2) = < 2, 3 >$; then L_n^1 has always two elements: $l_1^{n-1}2$ and $l_1^{n-1}3$. The next lists L_n^i with $i > 1$ are obtained as follows:

- consider the i-th element of $\mathcal{L}_{n-1}$ (i.e. l_i^{n-1});
- consider the list of the successors of the rightmost digit of l_i^{n-1} shifted starting from the rightmost digit of the rightmost element of L_n^{i-1} (i.e. $s(\overrightarrow{l_i^{n-1}}, \overrightarrow{last(L_n^{i-1})})$);
- paste l_i^{n-1} with each element of the list $s(\overrightarrow{l_i^{n-1}}, \overrightarrow{last(L_n^{i-1})})$.

Let us construct for instance the list $\mathcal{L}_4$:

$\mathcal{L}_1 =< 2 >$;

$L_2^1 = 2 \circ s(2,2) = 2\circ < 2,3 >=< 22, 23 >$, then

$\mathcal{L}_2 =< 22, 23 >$;

$L_3^1 = 22 \circ s(2,2) = 22\circ < 2,3 >=< 222, 223 >$;

$L_3^2 = 23 \circ s(3,3) = 23\circ < 3,4,2 >=< 233, 234, 232 >$, then

$\mathcal{L}_3 =< 222, 223, 233, 234, 232 >$;

$L_4^1 = 222 \circ s(2,2) = 222\circ < 2,3 >=< 2222, 2223 >$;

$L_4^2 = 223 \circ s(3,3) = 223\circ < 3,4,2 >=< 2233, 2234, 2232 >$;

$L_4^3 = 233 \circ s(3,2) = 233\circ < 2,4,3 >=< 2332, 2334, 2333 >$;

$L_4^4 = 234 \circ s(4,3) = 234\circ < 3,4,5,2 >=< 2343, 2344, 2345, 2342 >$;

$L_4^5 = 232 \circ s(2,2) = 232\circ < 2,3 >=< 2322, 2323 >$, then

$$\begin{aligned} \mathcal{L}_4 &= < 2222, 2223, 2233, 2234, 2232, 2332, 2334, 2333, 2343, \\ & \quad 2344, 2345, 2342, 2322, 2323 > \end{aligned}$$

We now prove the following:

Theorem 5.1.1 *Two consecutive elements of the list $\mathcal{L}_n$ differ only for one digit.*

Proof. We can proceed by induction on n:

base: if $n = 1$, then the theorem is trivially true since $\mathcal{L}_1 =< 2 >$;

inductive hypothesis: let us suppose that l_i^{n-1} and l_{i+1}^{n-1}, with $1 \leq i \leq |\mathcal{L}_{n-1}| - 1$, differ only for one digit;

inductive step: the list $\mathcal{L}_n$ is obtained by linking together the lists L_n^i for $i = 1, \ldots, |\mathcal{L}_{n-1}|$. Since the elements of each list L_n^i differ only for one digit by construction, we must prove the statement only for $last(L_n^i)$ and $first(L_n^{i+1})$, with $1 \leq i \leq |\mathcal{L}_{n-1}| - 1$.

Let J be the last element of $s(\overrightarrow{l_i^{n-1}}, \overrightarrow{last(L_n^{i-1})})$. Then we have:

$$last(L_n^i) = l_i^{n-1} J \ .$$

We also have:

$$L_n^{i+1} = l_{i+1}^{n-1} \circ s(\overrightarrow{l_{i+1}^{n-1}}, \overrightarrow{last(L_n^i)}) = l_{i+1}^{n-1} \circ s(\overrightarrow{l_{i+1}^{n-1}}, J) \ .$$

From the definition of the shifted list of the successors we deduce that the first element of a list $s(i, k)$ is always k, then:

$$first(L_n^{i+1}) = l_{i+1}^{n-1} J \ .$$

Since l_i^{n-1} and l_{i+1}^{n-1} differ only for one digit by the inductive step, this statement holds also for $last(L_n^i)$ and $first(L_n^{i+1})$. So, the theorem is proved.

□

At this point it is easily seen that $\overrightarrow{first(L_n^{i+1})}$ is a son of the second-last digit of $first(L_n^{i+1})$ and that $\overrightarrow{first(L_n^{i+1})} = \overrightarrow{last(L_n^i)}$. We remark that it is not possible that $\overrightarrow{last(L_n^i)}$ does not belong to the set of sons of the second-last digit of $first(L_n^{i+1})$, since from the definition of the *shifted production*, the construction we described above and the axiom of Ω_C (which is 2), we deduce that $\overrightarrow{last(L_n^i)} \in \{2, 3\}$, which are present in the production of each possible label.

The algorithm to generate $\mathcal{L}_n$

The aim is defining an algorithm which is not recursive for generating all the words of length n encoding the objects of size n. We base our procedure on the general idea that if a word c_j has been generated, then a single digit must be changed to generate the next word c_{j+1}, as the authors made in [BBGP].

The first word of the list is $w = 222\dots2$, where $w_i = 2$, for $i = 1, 2, \dots n$. The digit w_i to be modified at each step is determined using the algorithm of Walsh [Wa], i.e. using a $(n+1)$-dimensional array e, which is updated in such a way that, at each step, e_{n+1} points to w_i. Once w_i is determined, it can not be modified by simply increasing it by one [BBGP], but the definition of the *shifted production* must be taken in account. So, we use another array d (n-dimensional), which is defined as follows: $d_i = 0$ if w_i is modified according to the shifted production $s(k, 2)$; $d_i = 1$ if w_i is modified according to $s(k, 3)$. It is easy to prove that the introduction of the array d does not exchange the complexity of the recalled procedure of Walsh for generating Gray codes in $O(1)$ worst-case time per word [Wa]: his clever algorithm remains the starting point for the implementation of our method.

We note that d can also be used to establish when w_i is no more modifiable: from the definition of $s(k, j)$ it happens if $(d_i = 0 \wedge w_i = 3)$ or if $(d_i = 1 \wedge w_i = 2)$.

The generating procedure stops when the digit to be modified is w_1.

5.1.4 The case of Dyck paths

We consider now the specific class of Dyck paths. Each of them can be associated with a binary string according to the substitution, for example, of the up steps with the 1 bit and the down steps with 0. Let us consider a word of length n of the Gray code defined in Section 5.1.3. It has a correspondent Dyck path which, in turn, is associated with a binary string, both of length

$2n$ (in Section 5.1.4 we present an algorithm to directly translate a word in the associated binary string). We want to prove that, if we consider two consecutive binary strings corresponding to two consecutive words in the Gray code, they differ only for two bits (note that the *Hamming distance* between two binary strings encoding two Dyck paths is at least 2). For this aim we base on the ECO construction of Dyck paths [BDPP1]. We recall briefly its main features: if p is a Dyck paths of length $2n$ with the last descent of k steps, then it has $k+1$ active sites; we obtain each of its $k+1$ sons by inserting a peak in each active sites; the insertion of a peak in an active sites at hight h generates a Dyck path with $h+2$ active sites. Now we state the next proposition:

Proposition 5.1.1 *Two words of the Gray code differing for one digit correspond to binary strings which differ only for two bits.*

Proof. The last digit of a word denotes the number of active sites of the corresponding Dyck path, so if it is k, then the path has $k-1$ down steps in the last descent, according to the above mentioned ECO construction.

A Let us consider the case when the two words differs in the last digit. Let their codes be:

$$w_1 w_2 \dots w_i w_{i+1}$$

and

$$w_1 w_2 \dots w_i z_{i+1}.$$

We indicate a generic bit with the star $*$, so $w_1 \dots w_i$ corresponds to

$$\underbrace{1 * * \dots * * 1}_{2i-w_i+1} \underbrace{000 \dots 0}_{w_i-1}.$$

The adding of w_{i+1} corresponds to the insertion of a peak at height $w_{i+1}-2$ in the last descent of the Dyck path associated to $w_1 w_2 \dots w_i$.

So, the corresponding binary string is

$$\underbrace{1**\ldots**1}_{2i-w_i+1}\ \underbrace{00\ldots\ldots\ldots0}_{(w_i-1)-(w_{i+1}-2)}\ 1\underbrace{00\ldots0}_{w_{i+1}-1} = \underbrace{1**\ldots**1}_{2i-w_i+1}\ \underbrace{00\ldots0}_{w_i-w_{i+1}+1}\ 1\underbrace{00\ldots0}_{w_{i+1}-1} \tag{5.1}$$

(note that after the adding of w_{i+1}, the total number of bits is properly $2i+2$). In particular we have:

- in the case $w_{i+1} = w_i + 1$, when the peak is inserted in the active site with maximal height, the binary string becomes

$$\underbrace{1**\ldots**1}_{2i-w_i+1}1\underbrace{00\ldots0}_{w_i},$$

  in other words, the last ascent is longer than one step with respect to the Dyck path codified by the word $w_1w_2\ldots w_i$;

- in the case $w_{i+1} = 2$, when the peak is added at height 0 at the end of the Dyck path corresponding to $w_1w_2\ldots w_i$,the binary string is

$$\underbrace{1**\ldots**1}_{2i-w_i+1}\underbrace{00\ldots0}_{w_i-1}10.$$

In a similar manner, the addition of z_{i+1} after w_i transforms the corresponding binary string in

$$\underbrace{1**\ldots**1}_{2i-w_i+1}\ \underbrace{00\ldots0}_{w_i-z_{i+1}+1}\ 1\underbrace{00\ldots0}_{z_{i+1}-1}.$$

Let us suppose that $z_{i+1} = w_{i+1} + j$, where j can also assume negative values. If $j > 0$, then $j \in \{1, w_i + 1 - w_{i+1}\}$; if $j < 0$, then $j \in \{-1, 2 - w_{i+1}\}$. The word $w_1w_2\ldots w_iz_{i+1}$ corresponds to the binary string

$$\underbrace{1**\ldots**1}_{2i-w_i+1}\ \underbrace{00\ldots0}_{w_i-w_{i+1}+1-j}\ 1\ \underbrace{00\ldots0}_{w_{i+1}-1+j} \tag{5.2}$$

The difference between the words (5.1) and (5.2) is the location of the rightmost 1 bit, which in (5.2) is shifted of $|j|$ positions towards left

$(j > 0)$ or right $(j < 0)$ with respect to (5.1). It easily seen that the two strings differ only for the two bits in position w_{i+1} and $w_{i+1} + j$ from the right of the word.

B Let us consider now the case when the two words differ for two digits which are not the last ones:

$$w_1 w_2 \dots w_i w_{i+1} w_{i+2} \dots w_n \tag{5.3}$$

and

$$w_1 w_2 \dots w_i z_{i+1} w_{i+2} \dots w_n. \tag{5.4}$$

The associated binary strings after the insertion of w_{i+2} (i.e. the binary strings coding $w_1 \dots w_i w_{i+1} w_{i+2}$ and $w_1 \dots w_i z_{i+1} w_{i+2}$) are

$$\underbrace{1**\dots**1}_{2i-w_i+1}\ \underbrace{00\dots0}_{w_i-w_{i+1}+1}\ 1\ \underbrace{00\dots0}_{w_{i+1}-w_{i+2}+1}\ 1\underbrace{00\dots0}_{w_{i+2}-1}$$

and

$$\underbrace{1**\dots**1}_{2i-w_i+1}\ \underbrace{00\dots0}_{w_i-w_{i+1}+1-j}\ 1\ \underbrace{00\dots0}_{w_{i+1}-w_{i+2}+1+j}\ 1\underbrace{00\dots0}_{w_{i+2}-1}.$$

where, as in the preceding case, $z_{i+1} = w_{i+1} + j$. The insertions of the next digits w_k with $k = i+3, \dots, n$, which are equal in the two words, modify in the same way the last descent in the associated Dyck paths. Then, the difference between the two binary strings corresponding to them is not due to these insertions. So, also in this case, the binary strings related to (5.3) and (5.4) differ only for two bits.

□

From a binary string to the next one

The structure of the above proof can be used to derive an algorithm to generate a binary string p_{h+1} from the preceding one p_h, taking into account the generation order of the corresponding words in the Gray code. If u_h and u_{h+1} are two consecutive words in the Gray codes and p_h is the binary string corresponding to u_h, then:

- if u_h and u_{h+1} differ in the last digit and $j = \overrightarrow{u_{h+1}} - \overrightarrow{u_h}$ is the difference between these ones, then p_{h+1} is obtained from p_h by the shifting of $|j|$ positions of the rightmost 1 bit towards left if $j > 0$ or right if $j < 0$;
- if u_h and u_{h+1} differ in the i-th digit and j is the difference between the i-th digit of u_{h+1} and the i-th digit of u_h, then p_{h+1} is obtained from p_h by the shifting of $|j|$ positions of the second rightmost 1 bit towards left if $j > 0$ or right if $j < 0$.

The correctness of the above procedure can be easily checked and the algorithm is based on the proof of the preceding proposition.

From the word to the binary string

The proof of Proposition 5.1.1 suggests also the idea for an inductive algorithm which allows to derive the binary string corresponding to a given word in the Gray code. Let us suppose we have already encoded a word $w_1 \dots w_{n-1}$ in the binary string u. The adding of a new digit w_n modifies only the final part of u, as we can deduce from the first part of the proof of Proposition 5.1.1. In particular, the $w_{n-1} - 1$ rightmost 0 bits of u corresponding to the last descent of the related Dyck path, are replaced by $w_{n-1} + 1$ bits as in the following:

$$\underbrace{000\dots0}_{w_{n-1}-1} \longrightarrow \underbrace{\underbrace{000\dots0}_{w_{n-1}-w_n+1}\ 1\,\underbrace{000\dots0}_{w_n-1}}_{w_{n-1}+1}$$

It correspond to the adding of a peak in some site of the last descent of the Dyck path related to u.

Then, starting from the binary string 10 encoding the minimal Dyck path whose relating word in the Gray code is 2, it is possible to get the binary string corresponding to $w_1 \dots w_n$ from the knowledge of that one related to $w_1 \dots w_{n-1}$ by means of the following inductive procedure:

base: the binary string corresponding to the word 2 is 10;

inductive hypothesis: assume that u is the binary string codifying $w_1 \dots w_{n-1}$;

inductive step: then the binary string corresponding to $w_1 \dots w_n$ is obtained replacing the $w_{n-1} - 1$ rightmost 0 bits of u with the $w_{n-1} + 1$ bits $\underbrace{000\dots0}_{w_{n-1}-w_n+1}\ 1\underbrace{000\dots0}_{w_n-1}$.

In the following example the encoding of the word 2334 is shown:

$$\begin{array}{ccccccc} 10 & \rightarrow & 1100 & \rightarrow & 110100 & \rightarrow & 11011000 \\ (2) & & (23) & & (233) & & (2334) \end{array}$$

□

Note. The algorithm of Section 5.1.4 allows to find a binary string p_{h+1} starting from the preceding one p_j and the words u_h and u_{h+1} of the Gray code, corresponding to p_h and p_{h+1}, respectively. The algorithm of this section, whereas, generates the binary string from the corresponding word by means of an inductive procedure which can turn out too heavy for large values of n (the length of the word).

Hence, the preceding algorithm, having a low complexity, can be used to generate p_{h+1} in the case the string p_h and the words u_h and u_{h+1} are known.

5.1.5 Generalization to stable succession rules

The crucial point in the construction of the lists $\mathcal{L}_n$ is that each label k in the succession rule Ω_C has in its production the two labels 2 and 3, as we pointed out at the end of Section 5.1.3. This property, together with the definition of the *shifted production* of k, allows $last(L_n^i)$ and $first(L_n^{i+1})$ to be different only for one digit (which is not the last one). Starting from this remark, we generalize the procedure to define the Gray code to all those succession rules having a particularity similar to Ω_C which we would like to call *stability property*, meaning with this name that in each production of k we always find two labels, say c_1 and c_2, regardless of k.

Definition 2 *(stability property) We say that a succession rule* Ω

$$\Omega : \begin{cases} (a) \\ (k) \rightsquigarrow (e_1(k))(e_2(k)) \dots (e_k(k)) \;, \qquad k \in \mathbb{N} \end{cases}$$

is stable *if for each k there exist two indexes i, j $(i < j)$ such that $e_i(k) = c_1$ and $e_j(k) = c_2$ $(c_1 \leq c_2)$.*

We need also to extend the definition of shifted production for the labels of succession rules with the stability property, in order to obtain that each list of successors of any k ends with c_1 or c_2. We have the following *generalized shifted productions* of k, being $e_i(k) = c_1$ and $e_j(k) = c_2$:

$$\begin{cases} s(k, c_1) = < c_1, e_{i-1}(k), \dots, e_1(k), e_k(k), \dots, e_{j+1}(k), e_{j-1}(k), \dots, e_{i+1}(k), c_2 > \\ \\ s(k, c_2) = < c_2, e_{j+1}(k), \dots, e_k(k), e_1(k), \dots, e_{i-1}(k), e_{i+1}(k), \dots, e_{j-1}(k), c_1 > \;. \end{cases}$$

In Figure 5.2 we used two walks, very similar to the *factorial walks* on the integer half-line [B *et al.*], to visualize the generalized shifted production of k, the above one starting from c_1 and ending in c_2 (corresponding to $s(k, c_1)$) and the below one starting from c_2 and ending in c_1 (corresponding to $s(k, c_2)$).

Now, it is easy to prove that:

Proposition 5.1.2 *If Ω is a succession rule with the stability property, then the lists $\mathcal{L}_n$ defined by:*

$$\begin{cases} \mathcal{L}_1 = < a > \\ \\ \mathcal{L}_n = \Theta_{i=1}^{M} L_n^i \qquad \text{if} \quad n > 1 \end{cases}$$

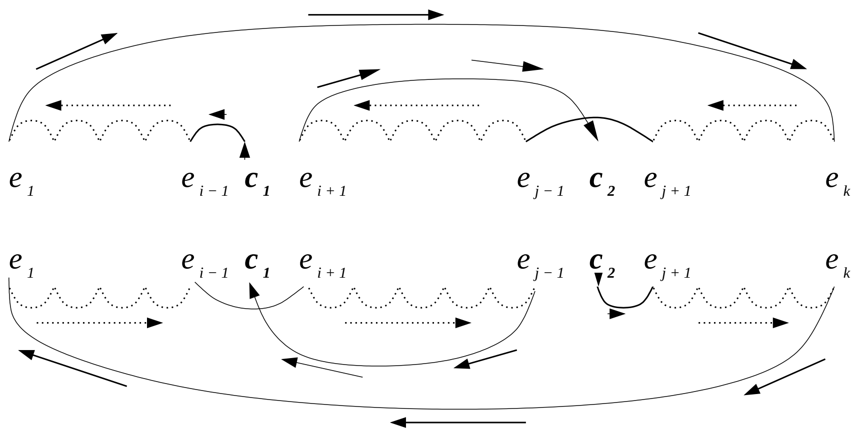


Figure 5.2 Generalized shifted production.

where $M = |\mathcal{L}_{n-1}|$ *and* L_n^i *is defined by*

$$\begin{cases} L_n^1 = l_1^{n-1} \circ s(\overrightarrow{l_1^{n-1}}, c_1) \\ \\ L_n^i = l_i^{n-1} \circ s(\overrightarrow{l_i^{n-1}}, \overrightarrow{last(L_n^{i-1})}) \quad if \quad i > 1 \end{cases}$$

form a Gray code in the sense of the definition of Section 5.1.1, where two consecutive words of length n differ for one digit (Hamming distance equals to one).

The proof is completely similar to that one of Theorem 5.1.1 and it is omitted.

Note that in the special case $i = 1,\ j = 2$ the generalized shifted production is:

$$\begin{cases} s(k, c_1) =< c_1, e_k(k), e_{k-1}(k), \ldots, e_3(k), c_2 > \\ \\ s(k, c_2) =< c_2, e_3(k), \ldots, e_k(k), c_1 > \ . \end{cases}$$

We now analyze some particular cases of succession rules with the stability property.

Example 1. Let us consider the following rule Ω_{F_o},

$$\Omega_{F_o}: \begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(3)(3) \ , \end{cases}$$

defining the odd Fibonacci numbers. It is easily seen that it satisfies the stability property, but the rule Ω_F,

$$\Omega_F: \begin{cases} (2) \\ (2) \rightsquigarrow (1)(2) \\ (1) \rightsquigarrow (2) \ , \end{cases}$$

defining Fibonacci numbers, does not satisfy the stability property. This is to say that such a property is not common to all the succession rules of a certain family (finite succession rules, in this case).

In the following examples it is shown that a similar behavior can be found also in factorial or transcendental rules.

□

Example 2. The factorial rule:

$$\Omega_M: \begin{cases} (1) \\ (k) \rightsquigarrow (1)(2)\dots(k-1)(k+1) \end{cases} \quad ,$$

defining the sequence of Motzkin numbers, does not satisfies the stability property, since only for $k \geq 3$ each label has $c_1 = 1$ and $c_2 = 2$ in its production. But the rules Ω_A of kind

$$\Omega_A: \begin{cases} (a) \\ (k) \rightsquigarrow (a)(a+1)\dots(k)(k+1)(k+d_1)\dots(k+d_m) \end{cases}$$

(with $a \geq 2$, $m = a-2$, $d_i \geq 0$ and $d_i \leq d_{i+1}$) are factorial and stable rules, with $i = 1$, $j = 2$, $c_1 = a$ and $c_2 = a+1$. The following well-known succession rule Ω_t, related to the Gray structure of the t-ary trees [BDP], is a particular case:

$$\Omega_t : \begin{cases} (t) \\ (k) \rightsquigarrow (t)(t+1)\dots(k-1)(k)(k+1)\dots(k+t-2)(k+t-1) \end{cases}$$

and the generalized shifted production is:

$$\begin{cases} s(k,t) =< t, k+t-1, k+t-2, \dots, k+1, k, k-1, \dots, t+2, t+1 > \\ \\ s(k,t+1) =< t+1, t+2, \dots, k-1, k, k+1, \dots, k+t-2, k+t-1, t > \end{cases} .$$

In the following, we present the construction of the list $\mathcal{L}_3$ in the case $t = 3$ in the above succession rule Ω_t.

$\mathcal{L}_1 =< 3 >$;

$L_2^1 = 3 \circ s(3,3) = 3\circ < 3, 5, 4 >=< 33, 35, 34 >$, then

$\mathcal{L}_2 =< 33, 35, 34 >$;

$L_3^1 = 33 \circ s(3,3) = 33\circ < 3, 5, 4 >=< 333, 335, 334 >$;

$L_3^2 = 35 \circ s(5,4) = 35\circ < 4, 5, 6, 7, 3 >=< 354, 355, 356, 357, 353 >$;

$L_3^3 = 34 \circ s(4,3) = 34\circ < 3, 6, 5, 4 >=< 343, 346, 345, 344 >$, then

$$\mathcal{L}_3 =< 333, 335, 334, 354, 355, 356, 357, 353, 343, 346, 345, 344 > .$$

If $t = 2$, then we find the succession rule Ω_C for Catalan numbers, enumerating, among other things, the binary trees. In [V] the author proposes a constant time algorithm for generating binary trees Gray codes. We note that our procedure, combined with the results of Section 5.1.4, is an alternative approach for this aim.

□

Example 3. Another particular case of Ω_A is the following family:

$$\Omega_r : \begin{cases} (r) \\ (k) \rightsquigarrow (r)(r+1)\dots(k)(k+1)^{r-1} \end{cases},$$

with $r \geq 2$. They satisfy the stability property, too, with $i = 1$, $j = 2$, $c_1 = r$ and $c_2 = r + 1$. If $r = 3$, then Ω_r is the well-known succession rule defining the sequence of Schröder numbers. The following rule Ω_s also codes the construction of Schröder paths, 2-colored parallelogram polyominoes, (4231, 4132)-pattern avoiding permutations, (3142, 2413)-pattern avoiding permutations [BDPP4, W2, W4] (these latter patterns are also considered in [BBL] for pattern matching decision problem for permutations).

$$\Omega_s : \begin{cases} (2) \\ (k) \rightsquigarrow (3)(4)\dots(k)(k+1)^2 \end{cases}$$

In this case it is $c_1 = 3$, $c_2 = 4$ and the associated shifted production is:

$$\begin{cases} s(k,3) =< 3, (k+1)_2, (k+1)_1, k, k-1, \dots, 5, 4 > \\ \\ s(k,4) =< 4, 5, \dots, k, (k+1)_1, (k+1)_2, 3 > \end{cases},$$

where the indexes differentiate labels with the same value. Note that $s(k_2, *) = s(k_1, *)$ ($* = 3$ or 4). The construction of the list $\mathcal{L}_3$ is:

$\mathcal{L}_1 =< 3 >$;

$L_2^1 = 3 \circ s(3,3) =< 33, 34_2, 34_1 >$, then

$\mathcal{L}_2 =< 33, 34_2, 34_1 >$;

$L_3^1 = 33 \circ s(3,3) =< 333, 334_2, 334_1 >$;

$L_3^2 = 34_2 \circ s(4_2, 4_1) =< 34_24_1, 34_25_1, 34_25_2, 34_23 >$;

$L_3^3 = 34 \circ s(4,3) =< 343, 345_2, 345_1, 344 >$, then

$$\mathcal{L}_3 =< 333, 334_2, 334_1, 34_24_1, 34_25_1, 34_25_2, 34_23, 343, 345_2, 345_1, 344 > .$$

□

Example 4. Succession rules of kind:

$$\Omega_B : \begin{cases} (r) \\ (k) \rightsquigarrow (b)^l(a)(a+1)\ldots(k)(k+d_1)\ldots(k+d_m) \\ (k) \rightsquigarrow (b)^k & if\ (k < a) \wedge (k \leq l) \\ (k) \rightsquigarrow (b)^{(k-1)}(a) & if\ k < a \quad , \end{cases}$$

with $l \geq 2$, $b < a$, $m = a - l - 1$, satisfy the stability property with $i = 1$, $j = 2$ and, denoting $b^l = b_1b_2\ldots b_l$, $c_1 = b_1$, $c_2 = b_2$. A well-known particular case is

$$\Omega_{GD} : \begin{cases} (2) \\ (2) \rightsquigarrow (3)(3) \\ (3) \rightsquigarrow (3)(3)(4) \\ (k) \rightsquigarrow (3)^2(4)\ldots(k)(k+1) \end{cases}$$

which encodes a construction for Gran Dyck paths [PPR]. The generalized shifted production associated is

$$\begin{cases} s(k, 3_1) =< 3_1, k+1, k, \ldots, 4, 3_2 > \\ \\ s(k, 3_2) =< 3_2, 4, \ldots, k, k+1, 3_1 > \quad . \end{cases}$$

The list $\mathcal{L}_3$ is obtained as follows:

$\mathcal{L}_1 =< 2 >;$

$L_2^1 = 2 \circ s(2, 3_1) =< 23_1, 23_2 >$, then

$\mathcal{L}_2 =< 23_1, 23_2 >;$

$L_3^1 = 23_1 \circ s(3_1, 3_1) =< 23_13_1, 23_14, 23_13_2 >$;

$L_3^2 = 23_2 \circ s(3_2, 3_2) =< 23_23_2, 23_24, 23_23_1 >$, then

$$\mathcal{L}_3 =< 23_13_1, 23_14, 23_13_2, 23_23_2, 23_24, 23_23_1 > .$$

□

Example 5. It is possible to find some examples among the transcendental succession rules which are stable or not. The classical rule defining the factorial numbers, which describes the construction of the permutations of length n by inserting the element n in any active site of any permutation of length $n-1$, is not stable (its production is: $(k) \rightsquigarrow (k+1)^k$). On the contrary, the following one Ω_p, defining the same sequence, is stable:

$$\Omega_p = \begin{cases} (2) \\ (2k) \rightsquigarrow (2)(4)(6)\dots(2k)(2k+2)^k . \end{cases}$$

Stability property is satisfied since each label $(2k)$ generates in the first two positions labels (2) and (4). The associated generalized shifted production is:

$$\begin{cases} s(2k,2) =< 2, (2k+2)_k, (2k+2)_{k-1}, \dots, (2k+2)_1, 2k, 2k-2, \dots, 4 > \\ \\ s(2k,4) =< 4, 6, \dots, 2k-2, 2k, (2k+2)_1, (2k+2)_2, \dots, (2k+2)_k, 2 > \end{cases},$$

where the indexes are useful to distinguish different labels but with the same value. In order to illustrate the combinatorial placement of Ω_p we propose a probably new ECO construction for the permutations which can be described by this rule. Let $\pi = \pi_1\pi_2\dots\pi_n$ be a permutation of S_n, we define an operator $\vartheta : S_n \longrightarrow 2^{S_{n+1}}$ (the power set of S_{n+1}) working as follows ($n \geq 1$):

- let $\pi_1 = k$, then ϑ generates $2k$ permutations $\pi' \in S_{n+1}$ which are indicated by $\pi'^{(i)}$, with $i = 1, 2, \ldots, 2k$;
- the entries of $\pi'^{(i)}$ are:
  1. if $i = 1, 2, \ldots, k$, then:
     - $\pi_1'^{(i)} = i$;
     - the other entries are the same of π where the entry i is replaced by $n + 1$.
  2. if $i = k + 1, k + 2, \ldots, 2k$, then:
     - $\pi_1'^{(i)} \pi_2'^{(i)} = (\pi_1 + 1)j$, where $j = 1, 2, \ldots, k$;
     - the other entries are obtained as follows:
       - If $\pi_1 \neq n$, then let ρ be the sequence, with length $n - 1$, obtained by π deleting π_1 after it has been interchanged with $\pi_1 + 1$. The remaining entries of $\pi'^{(i)}$ are the same of ρ where the entry j is replaced by $n + 1$.
       - If $\pi_1 = n$, then let ρ be the sequence obtained from π by deleting π_1. The remaining entries of $\pi'^{(i)}$ are the same of ρ where the entry j is replaced by n.

Remark. permutations $\pi'^{(i)}$ with $i = 1, 2, \ldots, k$ start with an ascent, while permutations $\pi'^{(i)}$ with $i = k + 1, k + 2, \ldots, 2k$ start with a descent.

It can be easily proved that if $\pi' \in S_{n+1}$, then there exists a unique $\pi \in S_n$ such that $\pi' \in \vartheta(\pi)$ $(n \geq 1)$, then operator ϑ satisfies Proposition 2.1 of [BDPP1], which ensures that the family of sets $\{\vartheta(\pi) : \pi \in S_n\}$ is a partition of S_{n+1}, so that ϑ provides a recursive construction of the permutations of $S = \bigcup S_n$.

In Figure 5.3 the action of ϑ on two different permutations of S_6 (the first one starting with an entry different from $n = 6$) is illustrated. Permutations

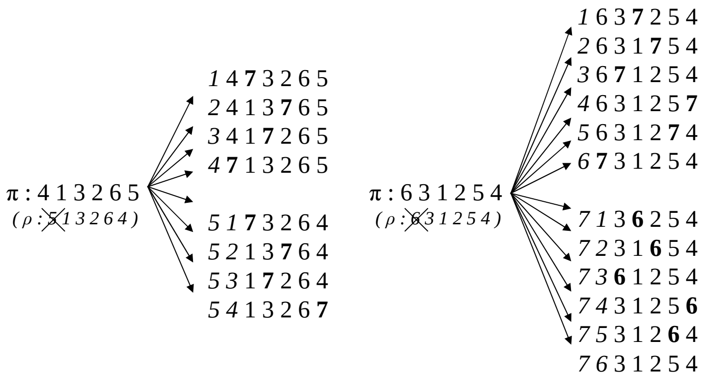


Figure 5.3 The action of ϑ on two different permutations of S_6.

$\pi'^{(i)}$, $i = 1, 2, \ldots, 2k$ generated by π by means ϑ are listed from the top to the bottom, being $\pi'^{(1)}$ at the top.

5.1.6 Conclusions and further developments

It is possible to find a lot of succession rules satisfying the stability property, but we are interested to the rules having some combinatorial relevance, as the ones presented in the above examples. In this way, with our procedure we are able to give a Gray code for the words (i.e. the paths whose nodes are the labels in the generating tree) encoding combinatorial Gray structures, i.e. those structures whose exhaustive generation can be described by a rule satisfying the stability property, which is not, as we have seen, an infrequent property.

Clearly, it would be better to have a Gray code for the objects instead of their encodes. Nevertheless, as we stated in Section 5.1.1, our procedure generates a Gray code which is not related to the nature of a particular class of combinatorial objects. Moreover, in some case it could be possible to translate the word of labels (the path in the generating tree) into the corresponding object. A further effort in this sense could be the research of algorithms for this translation in order to generalize the approach of Section 5.1.4 for Dyck paths. For this aim the ECO method can be useful, since by means of it each code is associated to a single object of the structure.

From the above examples it is possible to argue that the stability property of a succession rule does not depend on its "structural properties" , which have been discussed by the authors in [B *et al.*]. In the light of this fact, it is reasonable to ask if a stable succession rule can be considered as the representative, say *standard form*, of a set of rules which are all equivalent to it (two rules are said equivalent if they define the same number sequence [BDPR]). This is to say that the equivalence problem for succession rules could be amplified with respect to the investigation conducted in [BDPR] where the authors analyze the equivalence problem for some different kinds of rules: is it suitable the research of the set of rules equivalent to a stable succession rule?

Moreover, it is evident that it is not the sequence defined by the rule that induces it to be stable or not: factorial number sequence can be defined by a stable or not stable rule, as showed in Example 5. Consequently, a problem which naturally arises from this note is the existence of a succession rule with the stability property for any given number sequence. A first concerning question could be the following (to the authors knowledge the answer is open): is there a stable rule defining Motzkin numbers?

5.2 An exhaustive generation algorithm for Catalan objects

5.2.1 Preliminaries and notations

Here we give some notations we are going to use in the sequel, including Dyck paths which are described in a little bit different way (more suitable for our aim) with respect to Section 4.

We define *path* a sequence of points in $\mathbb{N} \times \mathbb{N}$ (they have never negative coordinates) and *step* a pair of two consecutive points in the path. A *Dyck path* is a path $\mathcal{D} := \{s_0, s_1, \ldots, s_{2n}\}$ such that $s_0 = (0,0)$ and $s_{2n} = (2n, 0)$, having northeast ($s_i = (x, y)$, $s_{i+1} = (x+1, y+1)$) or southeast ($s_i = (x, y)$, $s_{i+1} = (x + 1, y - 1)$) steps. The number of northeast steps is equal to the number of southeast steps and we denote *path's length* the number of its steps. In particular, $\mathcal{D}_n$ is the set of Dyck paths with length $2n$. In the sequel, if $\mathcal{D} \in \mathcal{D}_n$, then it has *size* n. A *peak* (resp.*valley*) is a point s_i such that step (s_{i-1}, s_i) is a northeast (southeast) and the step (s_i, s_{i+1}) is a southeast (northeast); moreover, we say *pyramid* p_h ,$\forall$ $h \in \mathbb{N}$, a sequence of h northeast steps following by h southeast steps such that if (s_i, s_{i+1}) is the first northeast step and (s_{i+2h-1}, s_{i+2h}) is the last southeast of this sequence, then $s_i = (x, 0)$ and $s_{i+2h} = (x+2h, 0)$. We also define *last descent* (*ascent*) the southeast (northeast) steps' last sequence of a Dyck path and we conventionally number its points from right (left) to left (right). Clearly the last point of last descent always coincides with last point of last ascent (see Figure 5.4). Moreover, if we say *height* $h(s_i)$ of a point s_i its ordinate and

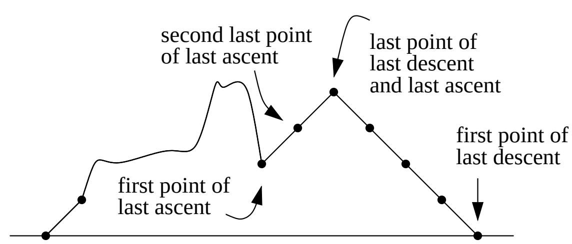


Figure 5.4 Numeration of points of Dyck path's last descendent and ascent.

non-decreasing point the extremity s_{i+1} of a northeast step (s_i, s_{i+1}), then we can define *area of a path* the sum of its non-decreasing points' heights and *maxima area path* P^n_{max} the pyramid p_n that contains, in geometric sense, all the paths of its size. Finally we call a path P "active" if we obtain another Dyck path when the first and the last step of P are taken off. This is equivalent to say that P does not have valleys with height $h = 0$.

Given a class of combinatorial objects $\mathcal{C}$ and a parameter $\gamma : \mathcal{C} \longrightarrow \mathbf{N}^+$ such that $\mathcal{C}_n = \{x \in \mathcal{C} : \gamma\ (x) = n\}$ is a finite set for all n, we define a *generating tree*. We assume there is only one element of minimal size in $\mathcal{C}$ and we describe the recursive construction of this set by using a rooted tree in which each node corresponds to an object. In particular, the vertices on the nth level represent the elements of $\mathcal{C}_n$, the root of the tree is the smallest element and the branch, leading to the node, encodes the choices made in the construction of the object. Starting from this idea and choosing the combinatorial class $\mathcal{D}$ of Dyck paths, we introduce another kind of generating tree which describe, fixed the size n, the recursive construction of $\mathcal{D}_n$. In the sequel we denote it with $\mathcal{D}_n$*-tree* which clearly has a finite number of levels and each object has the same size, regardless of the level.

5.2.2 Dyck paths

We define an operator which constructs $\mathcal{D}_n$. Since the cases $n = 1$ and $n = 2$ are trivial, we assume $n \geqslant 3$.

1. Consider P^n_{max} like the first path.
2. Take off the first and the last path's step and insert a peak in every point of the obtained path's last descent except for the last point. Every insertion generates a new Dyck path.
3. For each new generated path repeat the following actions until active paths are generated:
   - 3.1 take off the first and the last path's step
   - 3.2 insert a peak in every point of the obtained path's last descent. Every insertion generates a new path.

In Figure 5.5 we give an example of θ operator's action.

We prove that θ satisfies the following conditions:

Proposition 5.2.1

1. $\forall\ X_1\ ,\ X_2 \in \theta(P^n_{max})$, *then* $X_1 \neq X_2$*;*

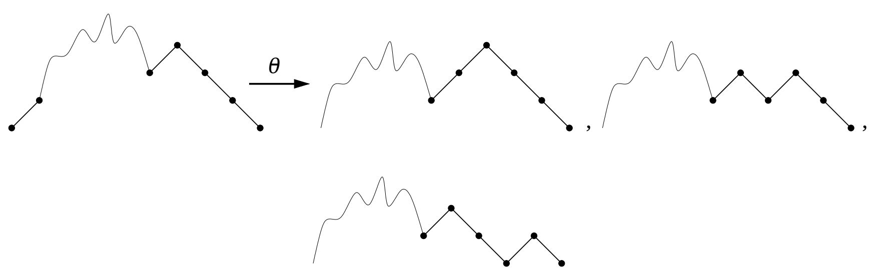


Figure 5.5

2. $\forall\ X_1\ ,\ X_2 \in \mathcal{D}_n$ *and* $X_1 \neq X_2$*, then* $\theta(X_1) \cap \theta(X_2) = \emptyset$.

Proposition 5.2.2 *$\forall\ Y\ \in\ \mathcal{D}_n\ \exists$ a finite succession $X_0, X_1, \ldots, X_k$ with $k \in \mathbb{N}$ and $X_k = Y$ such that :*

- $X_0 = P^n_{max}$;
- $X_{i+1} \in \theta(X_i) \qquad 0 \leq i \leq k-1.$

Proof Proposition 5.2.1. We prove point 2 since point 1 of the proposition is trivial. Consider $X_1\ ,\ X_2 \in \mathcal{D}_n$, $X_1 \neq X_2$ and divide both X_1 and X_2 in two parts as shown in Figure 5.6.

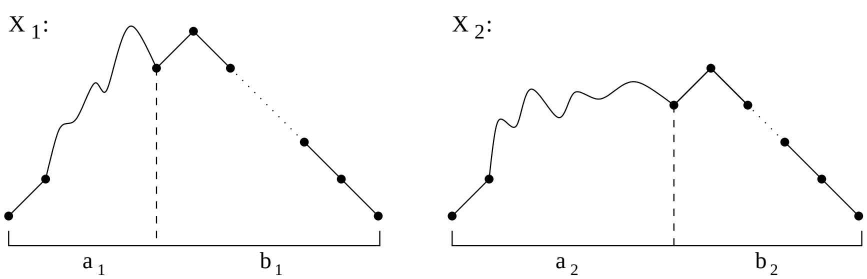


Figure 5.6

If $b_1 \neq b_2$, they remain distinct after the application of θ since it operates just on these parts. On the other hand if $b_1 = b_2$, then $a_1 \neq a_2$ and after the application of θ a_1 and a_2 remain different, then $\theta(X_1) \cap \theta(X_2) = \emptyset$ in both cases.

□

Proof Proposition 5.2.2. We consider a general path Y and we apply the inverse of θ operator on it; clearly θ^{-1} operator takes off the righter peak of Y and inserts a northeast step at the beginning of the path and a southeast step at the end. We have two possibilities:

1. The last ascent of Y has only one step, so in the obtained path $\theta^{-1}(Y)$ the peaks' number is reduced by one.

2. The last ascent of Y has at least two steps, so the number of last ascent's steps in $\theta^{-1}(Y)$ is reduced by one.

It is clear that after k times, for $k \in \mathbb{N}$, the number of peaks in $\theta^{-k}(Y)$ is one and $\theta^{-k}(Y) = P^n_{max}$.

□

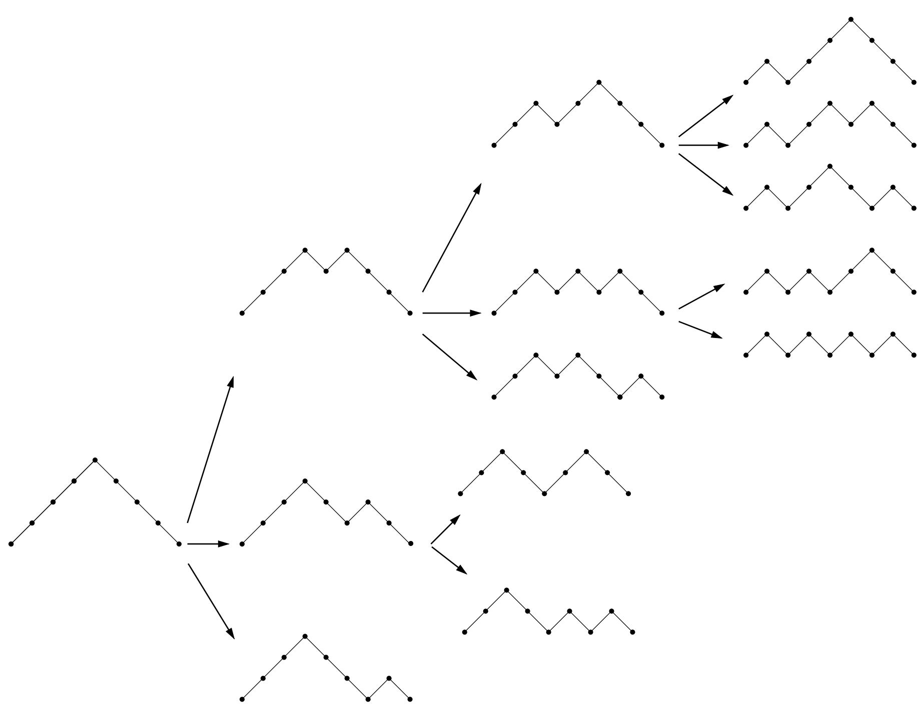


Figure 5.7 $\mathcal{D}_4$-tree.

We now describe θ's construction by using a rooted tree:

$\mathcal{D}_n$-tree ROOTED TREE

1. The root is P^n_{max} and it is at level zero;

2. if $X \in \mathcal{D}_n$-tree is at level $k \geq 0$ then $Y \in \theta(X)$ is a son of X and it is at level $k+1$.

In Figure 5.7 $\mathcal{D}_4$-tree is illustrated.

Theorem 5.2.1 *$\mathcal{D}_n = \mathcal{D}_n$-tree*

Proof. Given $X \in \mathcal{D}_n$-tree; it is clear that X is a Dyck path. Moreover, Proposition 5.2.1 assures there are not two copies of the same path in $\mathcal{D}_n$-tree $\Rightarrow |\mathcal{D}_n\text{-tree}| \leq |\mathcal{D}_n| \;\Rightarrow \mathcal{D}_n\text{-tree} \subseteq \mathcal{D}_n$.
Vice versa given $Y \in \mathcal{D}_n$, Proposition 5.2.2 assures that it is always possible to find a finite succession which joins P^n_{max} path to Y; so $Y \in \mathcal{D}_n$-tree since P^n_{max} is in $\mathcal{D}_n$-tree $\;\Rightarrow \mathcal{D}_n \subseteq \mathcal{D}_n$-tree.

□

Succession rule

We recall that, given a path P, it is $\theta(P) \neq \emptyset$ if and only if it is active, i.e. if it has not valleys with height $h = 0$. Moreover, from the definition of θ operator it is clear that the number of a path's sons is equal to the number of steps in its last descent. So, we have to label each path with an information recording the number of its sons and the height of its lowest valley. We use the following notation to connect the label of a parent P, having the height of its lowest valley equal to i, with the labels of its k sons:

$$(k, i) \hookrightarrow (c_1)(c_2)\dots(c_k).$$

Moreover, each of these k paths has the last descent with length s, with $s = 1, 2, \dots, k$. Now θ operator, after having taken off the first and the last step of P, inserts a peak in one of the last descent's point of the obtained path $\overline{P}$. This insertion increases the number of valleys in the generated path by one, with the exception of Y obtained by inserting the peak in the last point of $\overline{P}$'s last descent, since in this case, the path has the same number of valleys of its father P. Then, the height j of generated paths' lowest valley depends on the insertion of the peak. Indeed, if θ inserts the peak in the t-th point of the $\overline{P}$'s last descent with $1 \leqslant t \leqslant i-1$, then $j = t-1$, i.e. the lowest valley is generated by the peak insertion. On the other hand, if $i \leqslant t \leqslant k$, then $j = i-1$, i.e. the lowest valley is the same of $\overline{P}$. In Figure 5.8 we give an example of θ's action on a path with label $(3, 2)$. The production:

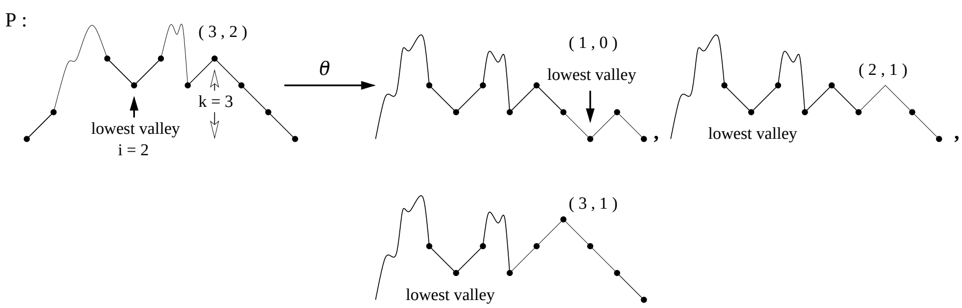


Figure 5.8

$$(k, i) \hookrightarrow (1, 0)(2, 1) \dots (i, i-1)(i+1, i-1) \dots (k, i-1).$$

We notice that the root of $\mathcal{D}_n$-tree does not have valleys and the second index of its label could be empty; nevertheless, we label the root by $(n-1, n-1)$ the same. Finally, the following succession rule is obtained:

$$\begin{cases} (n-1, n-1) \\ (k, i) \hookrightarrow (1, 0)(2, 1) \dots (i, i-1)(i+1, i-1) \dots (k, i-1) \end{cases}$$

The labels with $i = 0$ correspond to paths with at least a valley with height $h = 0$ and they do not generate any other path by θ operator. In Figure 5.9 an example of generating tree for $n = 5$ is shown.

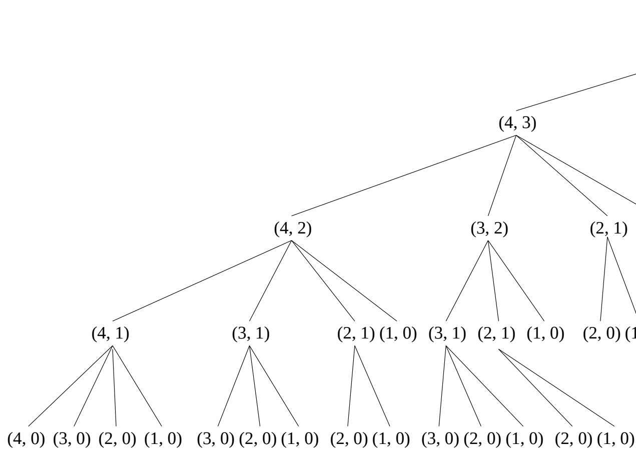


Figure 5.9

The generating algorithm

In the previous section θ operator is described by a rooted tree and $\mathcal{D}_n$'s paths are generated according to the $\mathcal{D}_n$-tree's levels. Nevertheless, we wish to find a method which sequentially lists the objects so that each one is generated only by the last generated path. This operation corresponds to visit all the nodes of $\mathcal{D}_n$-tree and for this reason it's helpful to order the sons of X path according to the decreasing length of their last descent so that the last one ends in p_1. In particular, the last P^n_{max}'s son is made by p_{n-1} followed by p_1. We name "firstborn" of a path P the son which has the longest last descent (In Figure 5.10 we give an example of a path's "firstborn").

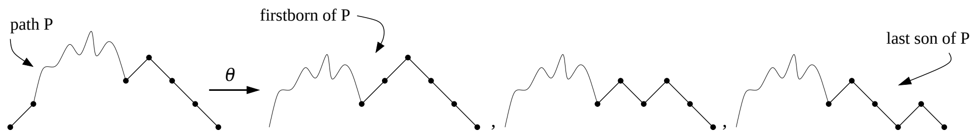


Figure 5.10

Clearly the "firstborn" of P^n_{max} can be generated simply overturning its peak. Then we generate all "firstborn" paths on the longest branch of $\mathcal{D}_n$-tree applying $(n-2)$ times the following operation:

op1: Take off the first and the last path's step, then insert a peak in the last point of the last descent (see Figure 5.11).

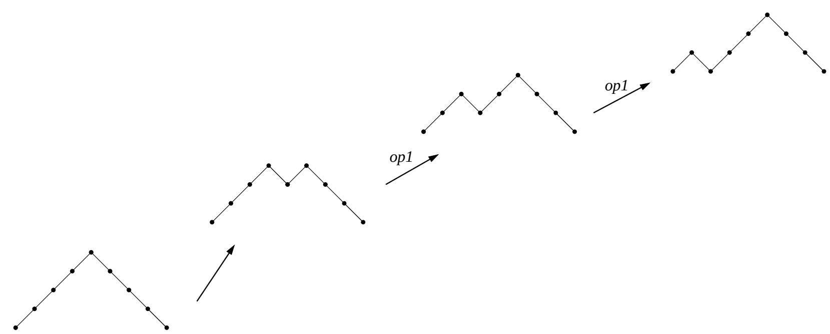


Figure 5.11

When *op1* is no more applicable, i.e. when we arrive at a leaf, we proceed to generate the leaf's brothers following the order given at the beginning of this subsection. So it is sufficient to apply the following operation on the last generated path:

op2: Overturn the rightmost peak in the path (see Figure 5.12) since, if

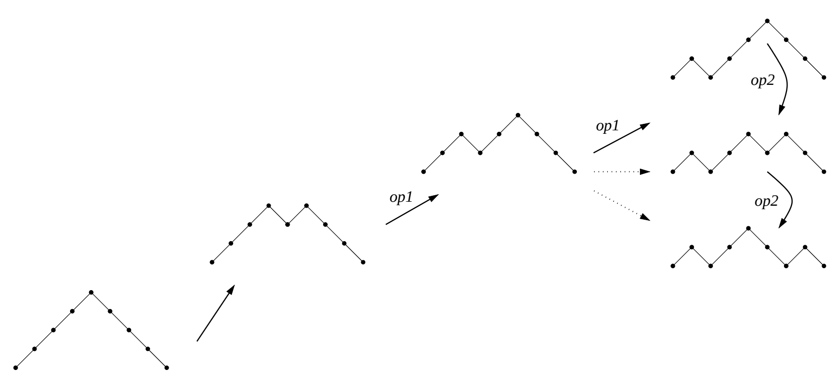


Figure 5.12

$Y_i \in \theta(X)$ with $1 \leqslant i \leqslant k-1$ and $k = |\theta(X)|$, then $op2(Y_i) = Y_{i+1} \in \theta(X)$. Indeed, Y_{i+1} is generated from X by means of θ taking off the first and the last step and inserting a peak in the $(k-i+2)th$ point of X's last descent; the generation of Y_{i+1} can be obtained also overturning the rightmost peak in Y_i.

After the last son of X is generated, we should go back to the $\mathcal{D}_n$-tree's preceding level, in other words we should pass to the immediately next brother of X, if it exists. We use the following operation to generate the "uncle" of last obtained path:

op3: Take off the rightmost p_1; then insert a northeast step at the beginning of the path and a southeast step in the second-last point of the last ascent (see Figure 5.13).

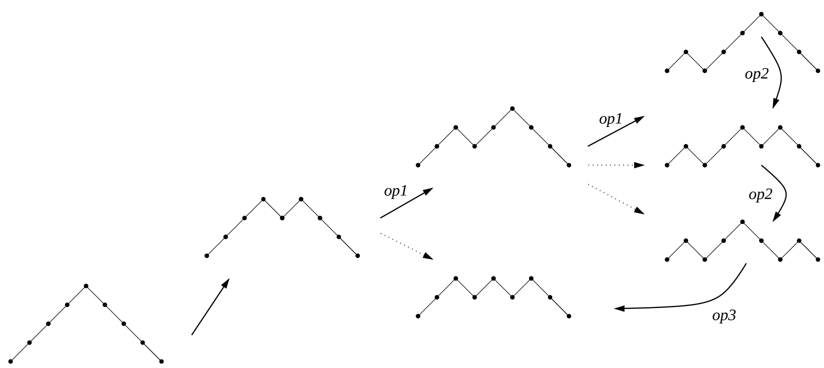


Figure 5.13

Op3 allows us to pass from a path ending in p_1 to its "uncle"; this fact it's very important because we can pass to another subtree of $\mathcal{D}_n$-tree, where we can apply *op1* and *op2* again.

The effects of *op3* on a path P ending in p_1 are illustrated in Figure 5.14.

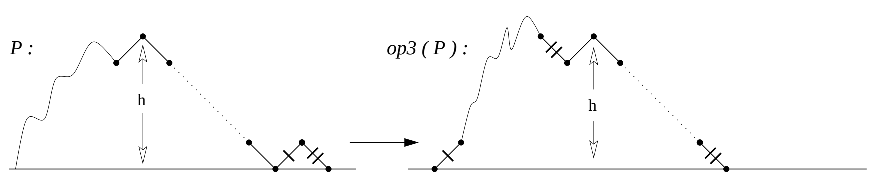


Figure 5.14

Let P the last son of a path P_i; the path P_{i+1}, P_i's brother, is obtained simply overturning its last peak (see Figure 5.15).

As we can see, the second path in Figure 5.14 is equivalent to P_{i+1} so $op3(P) = P_{i+1}$.

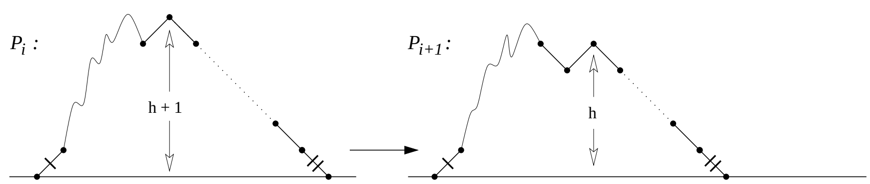


Figure 5.15

From their definition *op1*, *op2* and *op3* form a method to visit all the nodes of $\mathcal{D}_n$-tree and so, they generates all $\mathcal{D}_n$ paths.

We summarize the above arguments by means of the following algorithm:

Algorithm 1

$start\ \ with\ \ P^n_{max}$;

$generate\ \ the\ \ firstborn\ \ son\ \ of\ \ P^n_{max}\ \ overturning\ \ its\ \ peak$;

$P := firstborn\ \ son\ \ of\ \ P^n_{max}$;

while $P \neq\ the\ \ last\ \ son\ \ of\ \ P^n_{max}$ **do**

  if it's possible **then**

    $P' := op1(P)$

  else if it's possible **then**

    $P' := op2(P)$

  else

    $P' := op3(P)$

  end if;

  $P := P'$;

end while

Remark. Observing Figure 5.16 we can notice that it's possible to have more consecutive operations of the same kind but in particular we can have at most two consecutive applications of *op3*. Indeed we can have only two possibilities:

a) The path ends in p_1 which is preceded by a peak with height

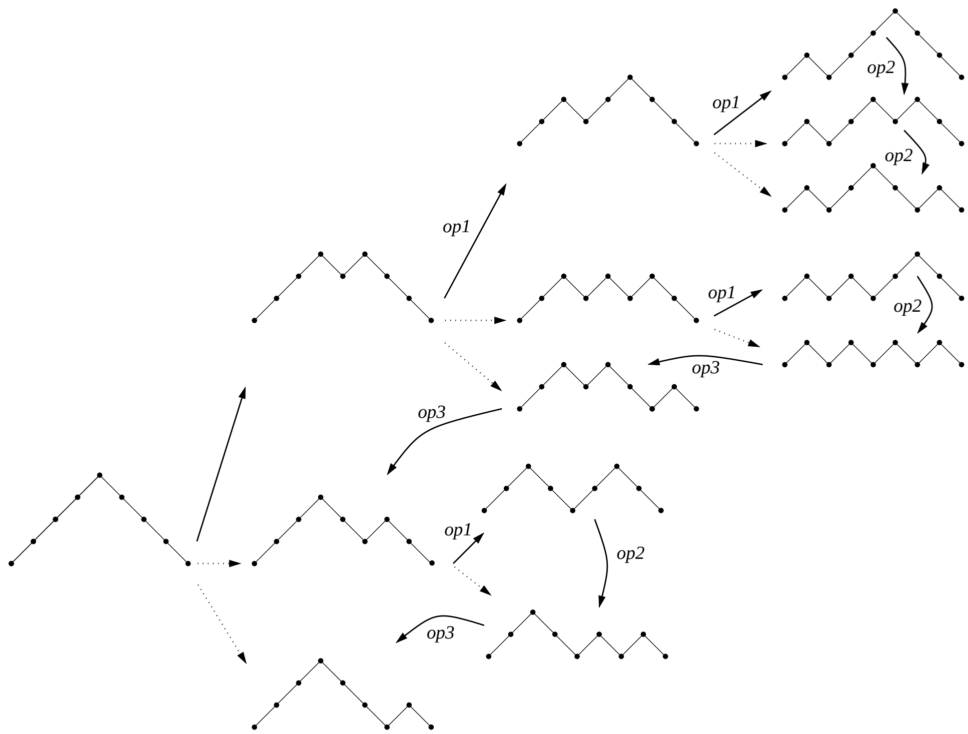


Figure 5.16

h ≥ 2

Op3 works only one time because its application, as we can see in Figure 5.17, generates a path that has the last peak with height h ≥ 2.

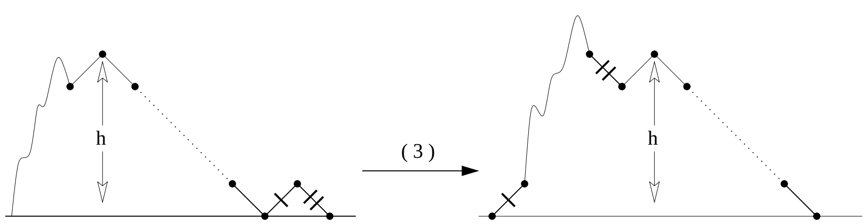


Figure 5.17

b) The path ends in at least two p_1

In this case the application of *op3* generates a path that ends again in p_1; we are in case **a)** and the application of *op3* is possible only another time (see Figure 5.18).

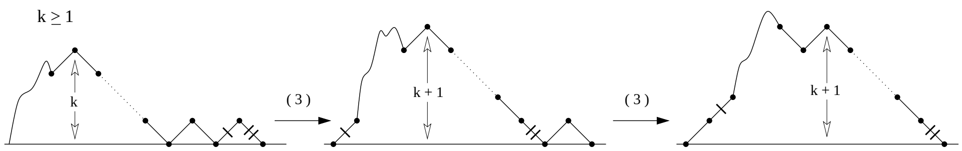


Figure 5.18

Analysis of Algorithm 1

Our aim is to realize a method which maintains constant the number of mean operations while each object in $\mathcal{D}_n$ is generated. If we associate to each path a binary word by coding with 1 a northeast step and with 0 a southeast, then it's clear that the three operations are characterized by a constant number of actions which exchange steps in the path. Indeed, we represent the word by a circular array where the last position is followed by the first one; we introduce a pointer to the first position of the array which always corresponds to the first step of the path (see Figure 5.19).

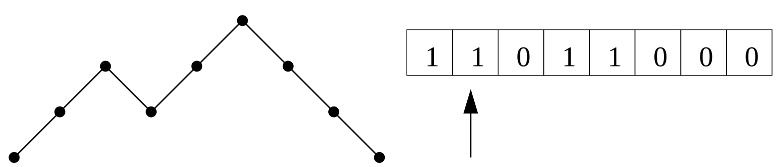


Figure 5.19

Op1 is equivalent to exchange the first bit 1 of the path with the first bit 0 of its last descent and then to move forward the pointer one position (the action of *op1* on the array is illustrated in Figure 5.20).
Op2 is equivalent to exchange the bits of the last sequence 10 in the array, while the pointer doesn't move (see Figure 5.21).
Finally, *op3* is equivalent to exchange the bits of the last and second-last pairs 10 and then to move backward the pointer one position (see Figure 5.22).

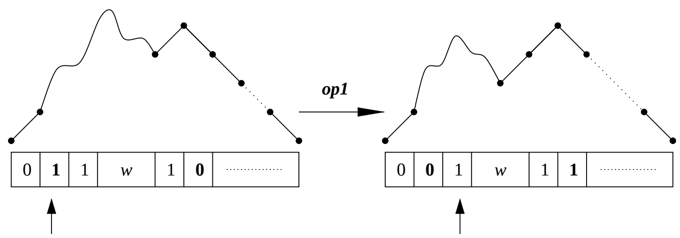


Figure 5.20

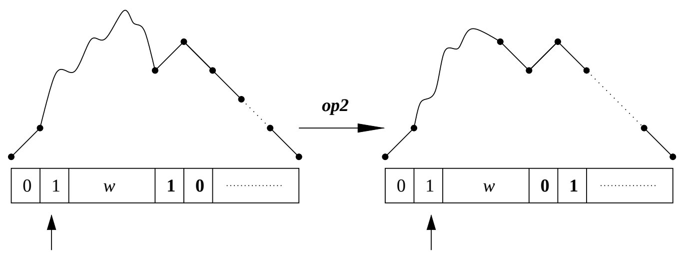


Figure 5.21 Action of *op2* operation on the array.

It's clear that the three operations require a constant number of actions independently of the length of the paths and **Algorithm 1** is a constant amortized time (CAT) algorithm.

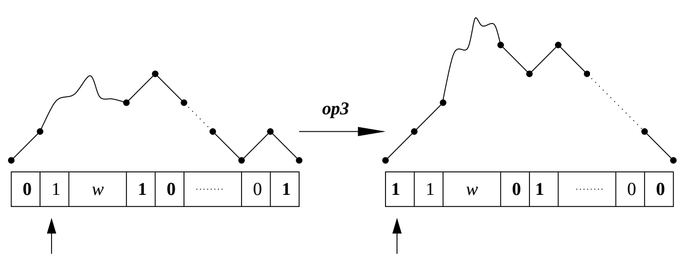


Figure 5.22 How word of bits changes by *op3*.

5.2.3 Conclusions

The practical advantages of our method are that it uses directly the combinatorial objects and it generates all the paths $\in \mathcal{D}_n$, with fixed n, without using the objects with smaller size.

Our studies have proved that the basic idea of this algorithm allows to obtain similar results for other classes of paths like Grand Dyck ($\mathcal{G}_n$) and

Motzkin ($\mathcal{M}_n$) paths; indeed, it's possible to obtain all the paths of $\mathcal{G}_n$ or $\mathcal{M}_n$ using operations very similar to *op1*, *op2* and *op3*.

Moreover, it is reasonable to think that this method could be applicable to other kinds of paths or to other combinatorial classes which are in bijection with the studied paths. For example we could study the classes of polyominoes or permutations enumerated by Catalan, Motzkin or Grand Dyck numbers (for definitions see for example.

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