Asymptotic tracking for constrained monotone systems

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Abstract

This paper addresses tracking control of nonlinear discrete-time monotone systems subject to input and state constraints. Forcing saturation on a previously designed controller may, in general, lead to destabilization or, at least, result in constraint violation and performance losses. Hereby it is shown that for a certain class of nonlinear monotone systems it is possible to design a static nonlinear output feedback which, saturated among suitable state-dependent bounds, is able to guarantee constraint satisfaction and asymptotic tracking of piecewise constant references, with a moderate on-line computational burden. Simulation experiments concerning a diffusion reaction process and the synthesis of a protein demonstrate the effectiveness of the proposed control strategy.

1 Introduction

Monotone systems [1]-[17] have recently attracted great attention in the control literature. So far most of the research efforts have been devoted to analysis issues while much less is known on specific control synthesis tools which could exploit system monotonicity in some respect. A crucial control problem is the design of an offset-free tracking controller for nonlinear systems subject to input and state constraints. In this respect, the ordering of trajectories in monotone systems could certainly help an efficient design of control laws that jointly ensure stability and constraint fulfilment. The present paper shows in fact that, for a certain class of nonlinear monotone systems, it is possible to design a stabilizing static output feedback in a straightforward way and then force saturation on the input taking into account state and input constraints. Based on this control design, it is possible to derive a tracking control algorithm which provides, with a moderate on-line computational burden, both constraint satisfaction and asymptotic tracking requirements. The paper is organized as follows. First a review of basic definitions and results on monotone systems is carried out in section 2. Section 3 first reviews preliminary results from [21] on tracking control of monotone systems subject to input constraints only, and then shows how for a monotone system it is possible to recast state constraints as appropriate state-dependent input constraints. Section 4 presents a control algorithm for monotone systems capable of handling state (in addition to input) constraints and analyzes its properties. In section 5 an heuristic modification of the control strategy is proposed for performance improvement. The applicability of the method and its effectiveness are illustrated by means of simulation examples in section 6. Finally some conclusions are drawn in section 7.

2 Notation and problem formulation

2.1 Notation

The notation used throughout the paper is mostly standard. $\mathbb{Z}_+ \triangleq \{0, 1, \ldots\}$ denotes the set of nonnegative integers and $\mathbb{R}^n_{\geq 0}$ the positive orthant in the n-dimensional Euclidean space. The constant unit signal will be denoted by $1(\cdot)$, defined as $1(k) = 1$ for all $k \in \mathbb{Z}_+$. In an Euclidean space $\mathbb{R}^\ell$ a partial order $\succeq$ induced by a positivity cone C is introduced. Let $C \subseteq \mathbb{R}^\ell$ be a nonempty, closed, convex, pointed ($C \bigcap -C = \{0\}$) cone with nonempty interior, then $v_1 \succeq v_2$ ($v_1, v_2 \in \mathbb{R}^\ell$) means that $v_1 - v_2 \in C$. Strict ordering is denoted by $v_1 \succ v_2$, meaning that $v_1 \succeq v_2$ and $v_1 \neq v_2$. Further, $v_1 \gg v_2$ if $v_1 - v_2$ is an interior point of C. The partial order is extended to signals $v(\cdot) : \mathbb{Z}_+ \to \mathbb{R}^\ell$ in the sense that $v_1(\cdot) \succeq v_2(\cdot)$ if $v_1(t) \succeq v_1(t)$ for all $t \in \mathbb{Z}_+$. Subsequently let us consider the order-theoretic notions $\inf(S)$ and $\sup(S)$ to indicate the greatest lower bound and, respectively, the least upper bound of a set [18]. Formally for subsets S of arbitrary partially ordered sets $(X, \ \preceq)$, the infimum of S is an element $w \in X$ such that

1. $w \preceq x, \quad \forall\, x \in S$
2. $\forall m \in X,$ if $m \preceq x \quad \forall\, x \in S,$ then $m \preceq w$

Similarly the supremum of S is an element $z \in X$ such that

1. $x \preceq z, \quad \forall\, x \in S$
2. $\forall m \in X,$ if $x \preceq m \quad \forall\, x \in S,$ then $z \preceq m$

2.2 Problem formulation

Consider the following discrete-time SISO nonlinear system

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

where $t \in \mathbb{Z}_+$, $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}$, $y(t) \in \mathbb{R}$, the map $f(\cdot, \cdot)$ is continuous in (x, u) and the map $h(\cdot)$ is continuous in x. The solution of (1) for the initial state $x(0) = x_0 \in \mathbb{R}^n$ and the input signal $u(\cdot) \triangleq \{u(k) \ : \ k \in \mathbb{Z}_+\} \in \mathcal{U}$ will be denoted, for all $t \geq 0$, by $\varphi(t, x_0, u(\cdot))$. With reference to the system (1), let C_x, C_u and C_y denote the order cones for the state, input and, respectively, output. Without loss of generality (otherwise, it is always possible to consider $-u$ as an input and/or $-y$ as an output), the considered order on the input and output spaces is $C_u = C_y = \mathbb{R}_{\geq 0}$.

Definition 1 - *The system (1) is said monotone if the following property holds, with respect to the orders on the state and the inputs for all $x_1, x_2 \in X$ and input signals $u_1(\cdot),\ u_2(\cdot) \in \mathcal{U}$:*

$$x_1 \succeq x_2 \text{ and } u_1(\cdot) \succeq u_2(\cdot) \Rightarrow \varphi(t, x_1, u_1(\cdot)) \succeq \varphi(t, x_2, u_2(\cdot)) \quad \forall t \geq 0 \tag{2}$$

and the output map $h(\cdot)$ is monotone with respect to the partial order on the state and output spaces, i.e. $h(\varphi(t, x_1, u_1(\cdot))) \geq h(\varphi(t, x_2, u_2(\cdot)))$ for all $t \geq 0$.

It is important to check monotonicity without computing the trajectories of (1). This amounts to checking monotonicity of the map $f(x, u)$ with respect to the partial orders in X and U. Remind that functions between ordered sets are said monotone if they preserve the given order [18].

Proposition 1 - *The system $x(t+1) = f(x(t), u(t))$ is monotone, with respect to the positivity cones C_x on the states and C_u on the input if and only if*

$$x_1 \succeq x_2 \text{ and } u_1 \succeq u_2 \Rightarrow f(x_1, u_1) \succeq f(x_2, u_2) \tag{3}$$

i.e. f is a monotone function of x and u.

In this paper, the control objective is that

1. the output $y(\cdot)$ track a piecewise constant reference $r(\cdot)$, i.e. a signal switching among different constant set-points;
2. the input $u(\cdot)$ satisfy the pointwise-in-time constraints

$$u(t) \in U \stackrel{\triangle}{=} \{u \ : \ \underline{u} \leq u \leq \bar{u}\},\ \forall t \in \mathbb{Z}_+ \tag{4}$$

3. the state $x(t)$ satisfy the pointwise-in-time constraints

$$x(t) \in S \subset X,\ \forall t \in \mathbb{Z}_+\,. \tag{5}$$

For the subsequent developments the following assumptions are made.

Assumption 1 - *For each constant set-point r there is associated a unique (state,input) equilibrium pair $(x_e(r), u_e(r))$ such that*

$$f(x_e(r), u_e(r)) = x_e(r), \qquad r = h(x_e(r)) \tag{6}$$

Assumption 2 - *It is assumed that the set of admissible states*

$$S \stackrel{\triangle}{=} \{x \ : \ \underline{g}_i \leq g_i(x) \leq \overline{g}_i,\ i = 1, 2, \cdots, m\} \tag{7}$$

is described by monotone constraints i.e. $g_i(x)$ are monotone functions of x.

Clearly the constraints (4) and (5) restrict the statically admissible set-points r to the ones that belong to the set

$$R \;=\; \{r:\; u_e(r) \in U,\; x_e(r) \in S\}. \tag{8}$$

In order to ensure convergence to the desired set-point, the reference $r(t)$ is further restricted to belong to the set

$$R_\delta \;=\; \{r : u_e(r) \in U,\; x_e(r) \in S_\delta\} \tag{9}$$

where

$$S_\delta \triangleq \{x \in S \;:\; x + w \in S,\; \forall w \;:\; \|w\|_\infty \leq \delta\}. \tag{10}$$

and $\delta > 0$ is an arbitrarily small number.

3 Some results on stabilization of monotone systems

In order to design a suitable tracking policy for (1) under the constraints (4) and (5), it is relevant to find how to stabilize such a system. A useful result on the stability of monotone systems is given hereafter.

Theorem 1 - *Suppose that:*

(i) *the dynamical system $x(t+1) = F(x(t))$ is monotone;*

(ii) *its trajectories are bounded for all $x(0) \in X$;*

(iii) *X contains exactly one equilibrium point x_e;*

(iv) *for every compact subset S of X, both $\inf(S)$ and $\sup(S)$ belong to X.*

Then x_e is asymptotically stable globally in X, i.e. it is stable and $\lim_{t\to\infty} \varphi(t, x_0) = x_e$ for all $x_0 \in X$. ■

In [19] this result was proved for continuous-time systems, but the same argument can be applied to discrete time systems as well. Notice that the above result refers to an unforced system $x(t+1) = F(x(t))$ and, hence, does not require that the open-loop system (1) have an unique and stable equilibrium. The steady-state behaviour of the system (1) will be useful in order to design a controller that meets the stated objectives.

Definition 2 - *Under the assumption 1, the system (1) admits a (possibly multi-valued) input to state (I/S) characteristic defined as follows*

$$k^X(u) \triangleq \{x \in X \;:\; f(x, u) = x\}\,. \tag{11}$$

If (1) admits an I/S characteristic, its input/output (I/O) characteristic is by definition the composition

$$k^Y(u) \triangleq \{y \in Y \;:\; y = h(x) \text{ and } f(x,u) = x\} \tag{12}$$

■

Our interest will be in the design of a static nonlinear output feedback $u(\cdot) = \ell(y(\cdot), r)$ meeting the control objectives stated in the previous section. In order to design a static nonlinear output feedback guaranteeing input constraint satisfaction the subsequent theorem, presented in [21] for continuous-time systems and rewritten here for discrete-time systems, will be useful.

Theorem 2 - *Suppose that the system (1) is monotone with respect to C_x in X, with $C_u = C_y = \mathbb{R}_{\geq 0}$, and that it has an I/O characteristic $k^Y(u)$. Moreover assume that $x(t+1) = f(x(t), \underline{u})$ and $x(t+1) = f(x(t), \overline{u})$ have bounded trajectories in X. Design an output feedback $u = \ell(y, r)$ with the following properties.*

1. *It admits, for each fixed r, only one intersection point in the plane (u, y) with the I/O characteristic $k^Y(u)$.*
2. *It is such that the closed-loop system*

$$x(t+1) = f(x(t), \ell(h(x(t)), r)) \tag{13}$$

is monotone with respect to the same partial order.

Then a saturated control law satisfying the property stated in point 1

$$sat(\ell(y(t), r)) = \begin{cases} \underline{u} & \text{if } \ \ell(y(t), r) < \underline{u} \\ \ell(y(t), r) & \text{if } \ \underline{u} \leq \ell(y(t), r) \leq \overline{u} \\ \overline{u} & \text{if } \ \ell(y(t), r) > \overline{u} \end{cases} \tag{14}$$

is such that the output asymptotically tracks any constant reference $r \in R$ globally in X, i.e. for all initial states $x_0 \in X$. ■

Proof - See the appendix.

The control law (14) is actually a static output feedback, i.e. it does not require measurement of the state if only input constraints (4) are present. Hereafter a sufficient condition for monotonicity preservation under feedback control is given.

Proposition 2 - *Given the monotone system (1) with $y = h(x_i)$, x_i being the i-th component of x, sufficient conditions for the closed-loop system under a stabilizing controller $u = \ell(h(x_i), r)$ to be still monotone are the following:*

1. $\dfrac{\partial f_i}{\partial u} > 0$ *and* $\dfrac{\partial f_j}{\partial u} = 0 \quad$ *for* $j \neq i, \quad \forall x \in X, \forall u \in U.$

2. $\dfrac{\partial f}{\partial x} \geq 0 \; \forall x \in X, \; \forall u \in U$ *i.e. its entries are non negative* $\forall x \in X, \; \forall u \in U$.

Proof - See the appendix.

An important issue concerns the ability of handling state constraints. In order to tackle this problem, the following theorem will be exploited for the subsequent developments.

Theorem 3 - *Suppose that the system (1) is monotone with respect to C_x in X, with $C_u = \mathbb{R}_{\geq 0}$, and that its trajectories are bounded in X. Moreover assume that the set of state constraints S satisfies assumption 2. Given $x \in S \subset X$, if the input satisfies the constraints*

$$u(\cdot) \in \mathcal{U}_x \stackrel{\triangle}{=} \{u(\cdot) : \underline{u}_x \leq u(t) \leq \overline{u}_x, \; \forall t \geq 0\}. \tag{15}$$

where $\underline{u}_x$ and $\overline{u}_x$ are constant inputs such that

$$\begin{aligned} \underline{u}_x &= \min\{u \; : \; \underline{g}_i \leq g_i(\varphi(t, x, u \; 1(\cdot))), \; \forall i, \; \forall t \geq 0\} \\ \overline{u}_x &= \max\{u \; : \; g_i(\varphi(t, x, u \; 1(\cdot))) \leq \overline{g}_i, \; \forall i, \; \forall t \geq 0\} \end{aligned} \tag{16}$$

then the constraints $\varphi(t, x, u(\cdot)) \in S$ are satisfied for all $t \geq 0$.

Proof - See the appendix.

This result will allow to face the tracking control problem considering appropriate state-dependent input constraints. Let us therefore introduce the state-dependent saturation

$$\mathrm{sat}_x(u) = \begin{cases} \max\{\underline{u}, \underline{u}_x\} & \text{if} \quad u < \max\{\underline{u}, \underline{u}_x\} \\ u & \text{if} \quad \max\{\underline{u}, \underline{u}_x\} \leq u \leq \min\{\overline{u}, \overline{u}_x\} \\ \min\{\overline{u}, \overline{u}_x\} & \text{if} \quad u > \min\{\overline{u}, \overline{u}_x\} \end{cases} \tag{17}$$

4 Tracking control algorithm

Based on the previous results, in particular theorems 2 and 3, it is now possible to formulate the tracking control algorithm described below.

Tracking Control (TC) algorithm - At time t, given the state $x(t)$ and the desired reference $r_d(t) \in R$, perform the following steps

1. Compute state-dependent bounds $\underline{u}_{x(t)}$ and $\overline{u}_{x(t)}$ according to (16).

2. Apply the control input
$$u(t) = \mathrm{sat}_{x(t)}(\ell(y(t), r_d(t))) \tag{18}$$

In order to study the convergence properties of the proposed algorithm, the excitability property, studied in the literature for continuous-time nonlinear systems [7], is next introduced for discrete time nonlinear systems.

Definition 3 - *A system is excitable if for any initial condition $x_0 \in X$ and any pair of input signals $u_1(\cdot) \succ u_2(\cdot)$, there exists $\tilde{t}$ dependent on $(u_1(\cdot), u_2(\cdot), x_0)$ such that*

$$\varphi(t, x_0, u_1(\cdot)) \gg \varphi(t, x_0, u_2(\cdot)), \quad \forall\, t > \tilde{t}. \tag{19}$$

If $\tilde{t}$ can be chosen independently of $(u_1(\cdot), u_2(\cdot), x_0)$ then the system is said uniformly excitable.

When the order in the input and in the state space is induced by an orthant it is possible to formulate a geometrical characterization of excitability in terms of the incidence graph of the system. This graphical characterization has been widely discussed in the literature for linear systems [13, 15] and for continuous-time nonlinear systems [7, 17, 22]. Here, we provide a result for discrete-time nonlinear systems. Following the same lines as in [7] we limit the class of systems for which the incidence graph is defined, to systems with strictly monotone interactions, i.e. a directed edge from the node i to the node j is defined only if

$$\hat{x}_i > \tilde{x}_i \Rightarrow f_j(\hat{x}, u) > f_j(\tilde{x}, u) \quad \forall \hat{x}, \tilde{x} \in X,\ \forall u \in U \tag{20}$$

where x_i indicate the i-th entry of the vector x. Similarly for the input node a directed edge from the input u to the node i is defined only if

$$\hat{u} > \tilde{u} \Rightarrow f_i(x, \hat{u}) > f_i(x, \tilde{u}) \quad \forall x \in X,\ \forall \hat{u}, \tilde{u} \in U \tag{21}$$

Theorem 4 - *Suppose that the system (1) is monotone with respect to the partial orders induced by the orthants $C_x \subset \mathbb{R}^n$ and $C_u \subset \mathbb{R}$ and it admits an incidence graph. Then the system is excitable if and only if each x_i is reachable from u through a directed path with a length not greater than n.*

Proof - See the appendix.

The definition of the following set and bounds is useful for the subsequent developments

$$\begin{aligned} &R_S \triangleq \{r \ :\ x_e(r) \in S\} \\ &\overline{u}_S = \sup\{u_e(r) \in U \ :\ r \in R_S\}, \quad \underline{u}_S = \inf\{u_e(r) \in U \ :\ r \in R_S\} \end{aligned} \tag{22}$$

Notice that $\underline{u} \leq \underline{u}_S \leq \underline{u}_{x(t)} \leq \overline{u}_{x(t)} \leq \overline{u}_S \leq \overline{u}$ for all $t \geq 0$. It can be shown that the proposed algorithm enjoys the following property.

Theorem 5 - *Let us assume that, for all $t \geq 0$, $r_d(t) = r_d \in R_\delta$, and that $x(t+1) = f(x(t), \underline{u}_S)$ and $x(t+1) = f(x(t), \overline{u}_S)$ admit a unique globally asymptotically stable equilibrium point in S. Under the assumptions of theorems 2, 3 and uniform excitability of the system in S, if $[\max\{\underline{u}, \underline{u}_{x(0)}\},\ \min\{\overline{u}, \overline{u}_{x(0)}\}] \neq \emptyset$, the tracking control algorithm guarantees that:*

(i) *the input and state constraints are satisfied for all $t \geq 0$;*

(ii) *the system asymptotically reaches the desired equilibrium i.e.* $\lim_{t\to\infty} x(t) = x_e(r_d)$, $\lim_{t\to\infty} u(t) = u_e(r_d)$ *and* $\lim_{t\to\infty} y(t) = r_d$.

Proof - See the appendix.

Remark 1 - *Under monotonicity of the map* $\ell(y(t), r)$ *with respect to* r*, the saturated control feedback in (18) is equivalent to the following one-step-ahead reference governor policy:*

1. *Solve:*

$$\begin{aligned} r(t) = \quad & \arg\min_{\overline{r}} (\overline{r} - r_d(t))^2 \\ & \textit{subject to} \\ & \max\{\underline{u}, \underline{u}_{x(t)}\} \leq \ell(y(t), \overline{r}) \leq \min\{\overline{u}, \overline{u}_{x(t)}\} \end{aligned} \tag{23}$$

2. *Apply the control input* $u(t) = \ell(y(t), r(t))$. ■

Remark 2 - *A feedback control law* $u = \ell(y, r)$ *satisfying the requirements that the two curves* $(u, k^Y(u))$*, i.e. the plant I/O characteristic, and* $(\ell(y, r), y)$*, i.e. the controller I/O characteristic, intersect just at one point* (u, y) *for each* r*, and preserving the monotonicity of the open-loop system, can be easily designed with a graphical procedure. In order to carry out easily a graphical choice of the feedback shape it is possible to re-parametrize the feedback as* $\ell(y, \theta(r))$ *for a suitable* r*-dependent parameter* θ*. Given the desired structure of* $\ell(y, \theta(r))$ *it is possible to determine* $\theta(r)$ *by solving the following algebraic equation*

$$k^Y(\ell(r, \theta(r)) = r \tag{24}$$

The existence of a suitable $\ell(y, r)$ *is ensured by assumption 1. Then, for each value of* r*, the computation of the matched value* $\theta(r)$ *is performed on-line, so that offset-free tracking is ensured.*
■

At time t the estimation of bounds $\underline{u}_{x(t)}$ and $\overline{u}_{x(t)}$ satisfying (16) deserves further discussion. In fact, these bounds must be computed on-line and it is, therefore, desirable to keep low the required computational burden. In order to limit the horizon for which (16) needs to be evaluated it is fundamental to know a time $\overline{t}$ such that for all $t > \overline{t}$ the state constraints will be fulfilled. For this purpose the input is further restricted to belong to the range $[\underline{u}_{S_\delta}, \overline{u}_{S_\delta}]$, where $\overline{u}_{S_\delta} = \sup\{u_e(r) \in U \ : \ r \in R_{S_\delta}\}$, $\underline{u}_{S_\delta} = \inf\{u_e(r) \in U \ : \ r \in R_{S_\delta}\}$ and $R_{S_\delta} \triangleq \{r : \ x_e(r) \in S_\delta\}$. For linear systems it is possible to determine $\overline{t}$ and then to perform easily an on line estimation of $\underline{u}_{x(t)}$ and $\overline{u}_{x(t)}$ as shown in the following theorem.

Theorem 6 - *Assume that the system*

$$x(t+1) = Ax(t) + Bu(t) \tag{25}$$

is asymptotically stable and monotone with respect to the partial orders induced by the orthants $C_x \subset \mathbb{R}^n$ *and* $C_u \subset R$. *Let the set of admissible states* $S \triangleq \{x : Mx \leq N\}$ *be a polytope described by monotone constraints and let the set of admissible inputs be* $U_{S_\delta} \triangleq [\underline{u}_{S_\delta}, \overline{u}_{S_\delta}]$. *Then there exists* $\overline{t}$ *such that for any* $x_0 \in S$ *and for any* $u \in U_{S_\delta}$, *the following implication holds:*

$$\varphi(t, x_0, u\ 1(\cdot)) \in S \ \ \textit{for}\ t = 0, 1, \ldots, \bar{t} \Longrightarrow \varphi(t, x_0, u\ 1(\cdot)) \in S,\ \forall t \geq 0. \tag{26}$$

Proof - See the appendix.

Remark 3 - *If* $[\max\{\underline{u}, \underline{u}_{x(0)}\},\ \min\{\overline{u}, \overline{u}_{x(0)}\}] \neq \emptyset$, *at time* k *given the state* $x(k) = x$, *it is possible to compute on-line* $\min\{\overline{u}, \overline{u}_{x(k)}\}$ *and* $\max\{\underline{u}, \underline{u}_{x(k)}\}$ *such that the state constraints are satisfied at any time instant, by solving the following linear programming problem*

$$\begin{array}{c} [\max\{\underline{u}, \underline{u}_{x(k)}\},\ \min\{\overline{u}, \overline{u}_{x(k)}\}] = [\min\limits_{u \in U_{S_\delta}} u,\ \ \max\limits_{u \in U_{S_\delta}} u] \\ \textit{subject to}\ x(t) = A^t x + \phi(t, 1)u \in S \quad \forall\ t \in [0,\ \overline{t}] \end{array} \tag{27}$$

where $\phi(t, 1) \triangleq \varphi(t, 0, 1(\cdot))$ *is the unit step response.* ■

Remark 4 - *For nonlinear systems the situation is theoretically more complicated. However, it is possible to estimate empirically the desired* $\overline{t}$ *by considering that* x *and* u *belong to compact sets. Then, on-line, it is possible to apply a bisection algorithm to find some feasible bounds on* u. *In order to reduce the computational burden, it is possible to compute off-line the bounds for a suitable number of state values in* S *belonging, for instance, to the segment joining the points* $\underline{x} = \inf(S)$ *and* $\overline{x} = \sup(S)$. *Then, exploiting the system's monotonicity and given the state* $x(t)$, *it is possible to obtain on-line some conservative information on the bounds. Moreover, it is convenient to exploit the bounds obtained at the previous step since they are guaranteed to be feasible and can, hopefully, be enlarged.* ■

5 Improved control strategy

Whenever the feasible steady-states are determined by the state constraints, the adopted control policy can be conservative. Less conservative approaches can be devised by adding degrees of freedom. For instance, enlarged saturation bounds $[\underline{u}_x - \Delta_u^-,\ \overline{u}_x + \Delta_u^+] \supset [\underline{u}_x,\ \overline{u}_x]$ could be found so as to guarantee the existence of a constant input that is feasible for the future time evolution. This approach is described hereafter.

1-Degree of Freedom Tracking Control (1DoF-TC) algorithm - At time t, given the state $x \in S$ and the desired reference $r_d(t) \in R$, perform the following steps.

1. Solve the following optimization problems:

$$\begin{array}{ccc}
\underline{u}_x - \Delta_u^- = \min\limits_{u\in[\underline{u}_S,\ \overline{u}_S],\ \Delta_u\geq 0} u - \Delta_u, & \overline{u}_x + \Delta_u^+ = \max\limits_{u\in[\underline{u}_S,\ \overline{u}_S],\ \Delta_u\geq 0} u + \Delta_u & \\
\text{subject to} & \text{subject to} & \\
\underline{u}_{x(0)} \geq u & \overline{u}_{x(0)} \leq u & (28)\\
u - \Delta_u \in U & u + \Delta_u \in U & \\
\hat{x}^- = f(x, u - \Delta_u) \in S & \hat{x}^+ = f(x, u + \Delta_u) \in S & \\
\varphi(t, \hat{x}^-, u\ 1(\cdot)) \in S \quad \forall\, t > 0 & \varphi(t, \hat{x}^+, u\ 1(\cdot)) \in S \quad \forall\, t > 0 &
\end{array}$$

   where

2. Then apply the control input

$$u(t) = \mathrm{sat}^1_{x(t)}(\ell(y(t), r_d(t))) \tag{29}$$

where $\underline{u}_{x(0)}$ and $\overline{u}_{x(0)}$ are estimated bounds satisfying (16) at time $t = 0$.

$$\mathrm{sat}^1_x(u) = \begin{cases} \underline{u}_x - \Delta_u^- & \text{if} \quad u < \underline{u}_x - \Delta_u^- \\ u & \text{if} \quad \underline{u}_x - \Delta_u^- \leq u \leq \overline{u}_x + \Delta_u^+ \\ \overline{u}_x + \Delta_u^+ & \text{if} \quad u > \overline{u}_x + \Delta_u^+ \end{cases} \tag{30}$$

Notice that it is possible to evaluate a time $\overline{t}$ such that for all $t > \overline{t}$ the state constraints will be fulfilled in a similar manner as discussed in theorem 6 and remark 4.

6 Simulation examples

The effectiveness of the proposed control procedure is now illustrated by means of two different numerical examples of practical interest.

6.1 Example 1

Diffusion reaction processes are described by equations that present spatial and temporal dependence. In order to handle these models, they are usually approximated through a spatial discretization subdividing the reactor in a cascade of cells with a length depending on the accuracy required by the model. Whenever the cells are all equal, the following linear model is obtained

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \tag{31}$$

$$\begin{aligned} A &= \begin{bmatrix} \beta - k & \gamma & 0 & 0 & \cdots & 0 \\ \alpha & \beta & \gamma & 0 & \cdots & 0 \\ 0 & \alpha & \beta & \gamma & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \gamma \\ 0 & \cdots & \cdots & 0 & \alpha & \beta + \gamma \end{bmatrix}, \\ B &= \begin{bmatrix} b_1 & 0 & 0 & \cdots & 0 \end{bmatrix}' \end{aligned} \tag{32}$$

for which the conservation relation $\alpha + \beta + \gamma = 0$ has been imposed and k denotes a dispersion coefficient acting on the first cell. For the simulation experiments we have used $n = 100$, $b_1 = 1$, $k = 0.1$, $\alpha = 2.3$, $\beta = -2.5$ and $\gamma = 0.2$. The structure of the system suggests the choice

$$y = x_1 \tag{33}$$

as output map. The considered system is positive and, hence, monotone with respect to the positive orthant. The I/O characteristic of the system is the line

$$y = \kappa u \quad (\kappa = 0.41\overline{6}) \tag{34}$$

In order to apply the proposed procedure the system has been discretized with sampling time $T_s = 0.01$ and a suitable control structure satisfying the conditions of theorem 2 is

$$u = -\theta_1 y + \theta_2 \tag{35}$$

Chosen the desired reference $r \in R$, for a given θ_1, the corresponding $\theta_2(r)$ is easily computed as

$$\theta_2(r) = \left(\frac{1}{\kappa} + \theta_1\right) r \tag{36}$$

The monotonicity condition of the closed loop system imposes the constraint $\theta_1 < \frac{1+T_s(\beta-k)}{T_s b_1} \simeq 97.39$, obtained by exploiting proposition 2; moreover, $\theta_1 = 97$ has been selected. The obtained behaviour of the proposed tracking strategy is shown by simulation experiments choosing the following input and state constraints

$$0.1 \le u \le 1, \quad 0 \le x_i \le 1 \quad \text{ for } i = 1,\ 2, \ldots, n \tag{37}$$

The statically admissible set-points are $0.0417 \le r \le 0.4167$. The existence of a unique equilibrium point is guaranteed for all $r \in R$. The output response to a square wave set-point, applying the control law $u = \text{sat}_x(-97y + \theta_2(r))$, is shown in figure 1. The input and, respectively, state responses are reported in figures 2 and, respectively, 3. Finally figure 1 also displays the choice of the selected reference r.

6.2 Example 2

An interesting application for the proposed approach is the n-dimensional cooperative system, which extends a model of the protein synthesis in the cell [17]

$$\begin{aligned} \dot{x}_1 &= -\alpha_1 x_1 + \gamma(x_n) + u \\ \dot{x}_i &= -\alpha_i x_i + x_{i-1} \qquad i = 2, \ldots, n \end{aligned} \tag{38}$$

where $\alpha_i > 0$ for all i, and $\gamma(x_n) = x_n^2/(1+x_n^2)$. For $u > 0$, the equilibria satisfy the relationships

$$\begin{aligned} x_i &= (\alpha_{i+1} \cdots \alpha_n)\, x_n, \quad i = 1, 2, \cdots, n-1 \\ \alpha x_n &= \gamma(x_n) + u \end{aligned} \tag{39}$$

where $\alpha = \alpha_1\ \alpha_2 \ \cdots \ \alpha_n$. For the simulation experiments the following parameter values have been selected

$$n = 5, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, \quad \alpha_5 = 0.55 \tag{40}$$

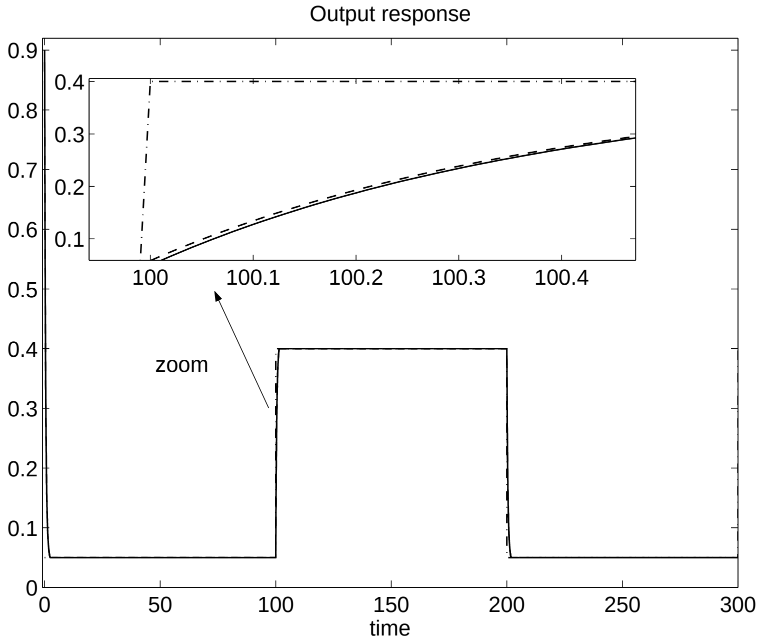

Figure 1: Desired reference (dashed-dot), feasible reference (dashed) and output response (solid)

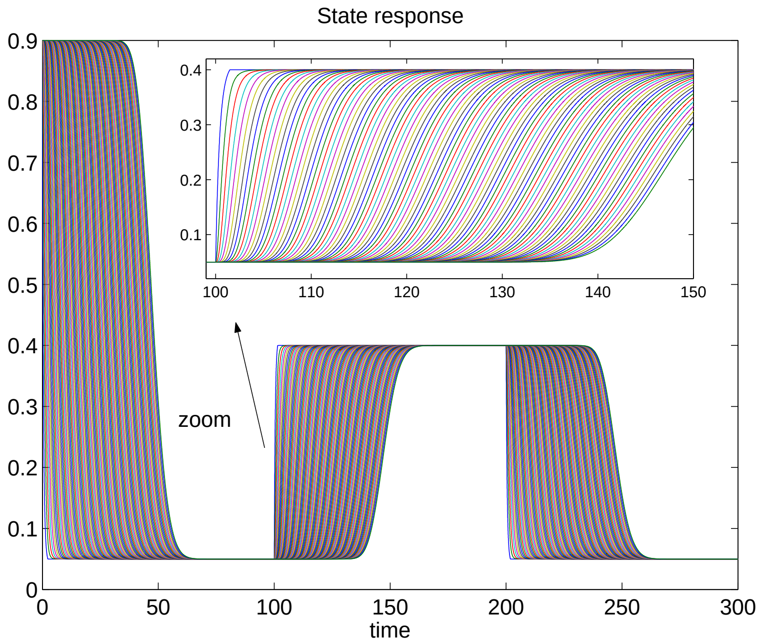

Figure 2: State response for $1 \leq i \leq n = 100$

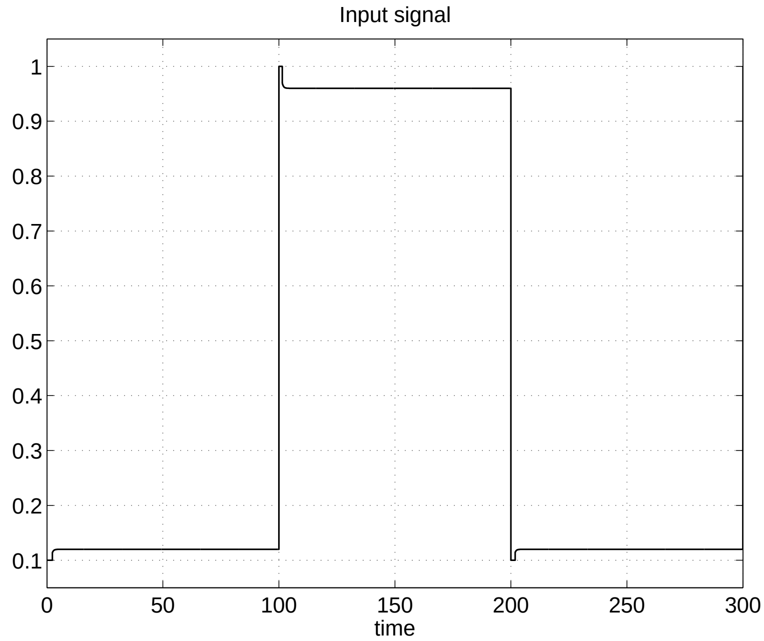

Figure 3: Saturated input

A possible choice for the output is

$$y = x_1 \tag{41}$$

Then the parametrized family of equilibrium points for (38) with respect to y is easily computed by (39). It is straightforward to check that the system (38) is monotone with respect to the order induced by the positivity cone $C_x = \mathbb{R}^n_{\geq 0}$ in $X = \mathbb{R}^n_{\geq 0}$. The I/O characteristic of the system (38), (41) is not well-defined. It is an hysteresis, as shown in figure 4, which presents multiple equilibria for some values of u. The system model has been discretized by the Eulero's technique with a sampling time $T_s = 0.01$. In order to stabilize the branch of unstable equilibrium points in $\mathbb{R}_{\geq 0}$, it is straightforward to design a controller satisfying the conditions of theorem 2. A suitable choice, according to theorem 5, is

$$u = -\theta_1 y^2 + \theta_2 \tag{42}$$

Chosen the desired reference $r \in R$, for a given θ_1, the corresponding $\theta_2(r)$ is the following

$$\theta_2(r) = \alpha_1 r + \theta_1 r^2 - \frac{r^2}{(\frac{\alpha}{\alpha_1})^2 + r^2} \tag{43}$$

The monotonicity condition of the closed loop system imposes the constraint $\theta_1 \leq \frac{1-\alpha_1 T_s}{8T_s} \simeq 12.3750$ by proposition 2. In order to get good performance, $\theta_1 = 12.37$ has been selected. The obtained behaviour of the proposed tracking strategy is shown by simulation experiments with the input and state constraints selected as

$$0.02 \leq u \leq 1, \quad 0 \leq x_i \leq 4 \quad \text{for } i = 1,\ 2, \ldots, 5 \tag{44}$$

The statically admissible set-points, induced by the input constraints, are $0.022 \leq r \leq 1.925$. The existence of a unique equilibrium point is guaranteed for all $r \in R = [0.022,\ 1.925]$. The

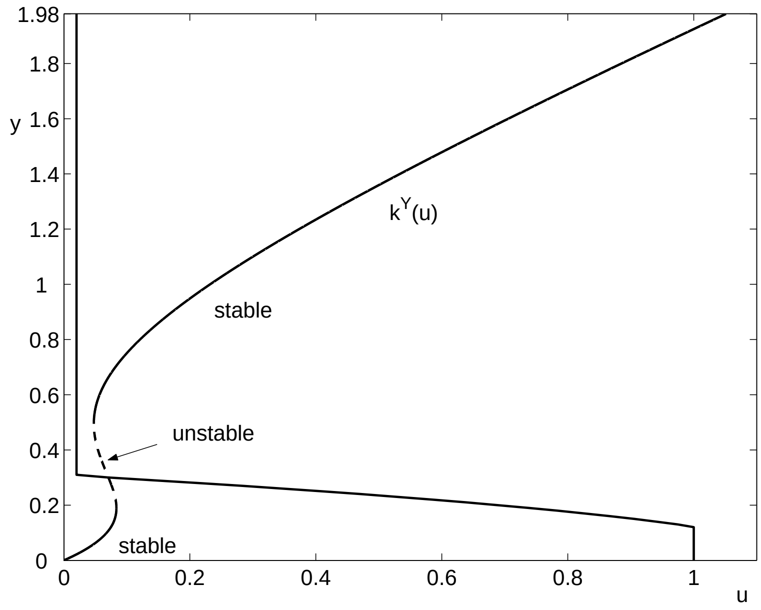

Figure 4: The intersection between the plant I/O characteristic $k^Y(u)$ and $u = \text{sat}(\ell(y, r))$ for $r = 0.3$

output response to a square wave set-point of amplitude ± 0.8 and offset 1.1, applying the control law $u = \text{sat}_x(-\theta_1 y^2 + \theta_2(r))$, is shown in figure 5. The applied input reported in figure 6 guarantees convergence to the desired reference and state constraint satisfaction as illustrated in figure 7. Notice how, at the beginning, the feasible inputs are restricted (figure 6) in order to guarantee state constraint fulfilment. It has been empirically estimated $\bar{t} = 400$ as a time window length for checking state constraints satisfaction. Moreover, since the positive orthant is invariant, if the interval $[\underline{u}_{x_0},\ \overline{u}_{x_0}]$ is not empty, then u_{min} is feasible and turns out to be a good initial point in the optimization problems (16).

In order to carry out a comparison between the behaviour of the tracking strategy and the 1DoF-TC algorithm, the following input and state constraints have been selected in the simulation experiments

$$0 \leq u \leq 2, \quad 0.02 \leq x_i \leq 2 \quad \text{for } i = 1,\ 2, \ldots, 5. \tag{45}$$

The statically admissible references, induced by the state constraints, are $0.02 \leq r \leq 1.1$ and the relative equilibrium states are $[0.02,\ 0.02,\ 0.02,\ 0.02,\ 0.0364]' \leq x_e(r) \leq [1.1,\ 1.1,\ 1.1,\ 1.1,\ 2]'$. The existence of a unique equilibrium point is guaranteed for all $r \in R = [0.02,\ 1.1]$. The obtained input bounds for the tracking control algorithm are $\underline{u}_S \simeq 0.019$ and $\overline{u}_S \simeq 0.3$. A comparison of the output response to a square wave set-point of amplitude ± 0.35 and offset 0.65, applying the control law $u = \text{sat}_x(-\theta_1 y^2 + \theta_2(r))$ and $u = \text{sat}_x^1(-\theta_1 y^2 + \theta_2(r))$, is shown in figure 8. The applied inputs reported in figure 9 guarantee convergence to the desired reference and state constraint satisfaction as illustrated in figure 10. Notice how, at the beginning, the feasible inputs are less restricted in the 1DoF-TC algorithm (figure 9) and a significative improvement is obtained, obviously at the expense of a superior computational burden.

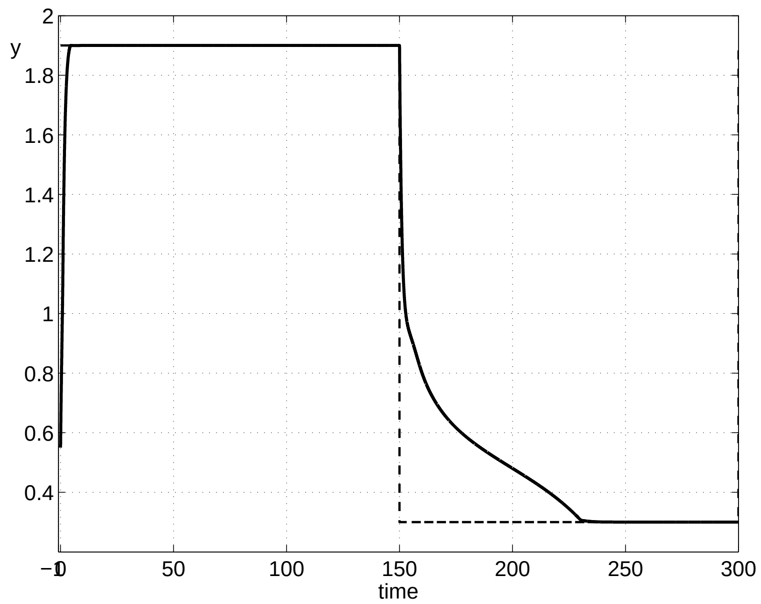

Figure 5: Desired output (dashed) and output response (solid)

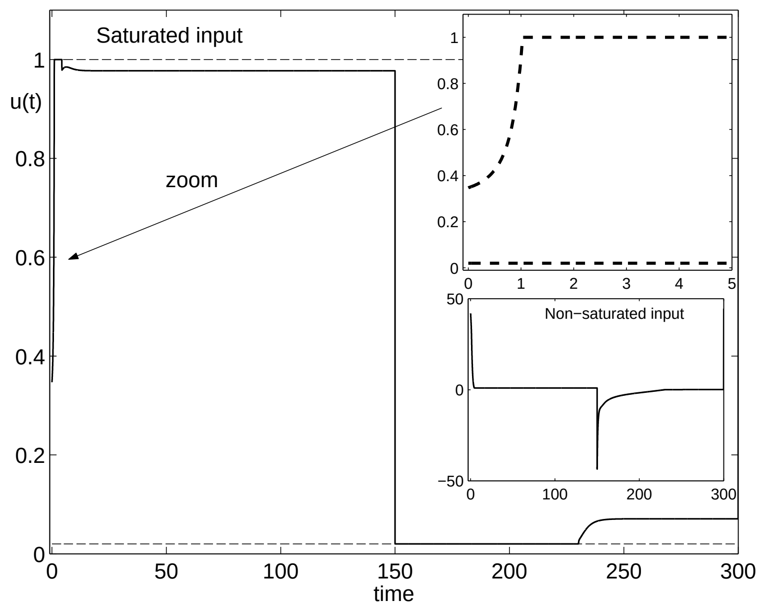

Figure 6: Saturated input (solid); behaviour of $\min\{\overline{u}, \overline{u}_{x(t)}\}$ and $\max\{\underline{u}, \underline{u}_{x(t)}\}$ (dashed) in the smaller picture on the top and non-saturated input (solid) in the smaller picture on the bottom.

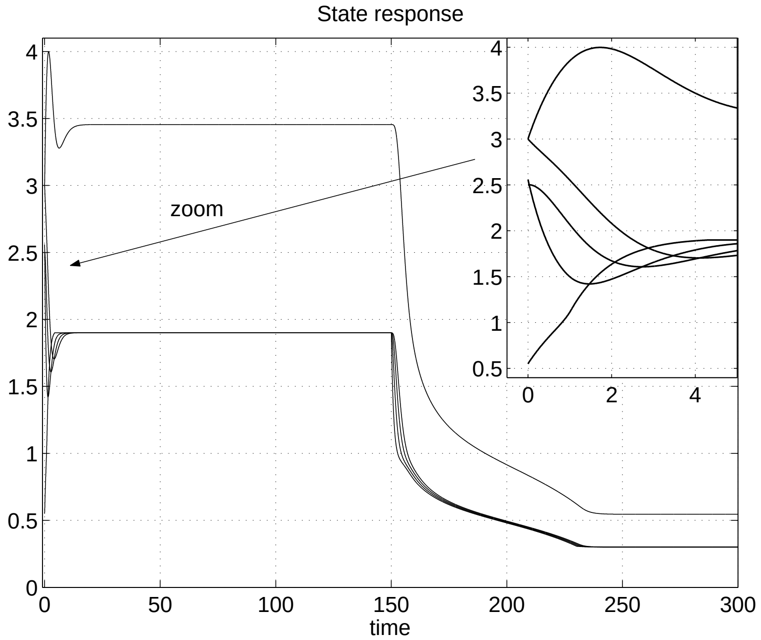

Figure 7: State response

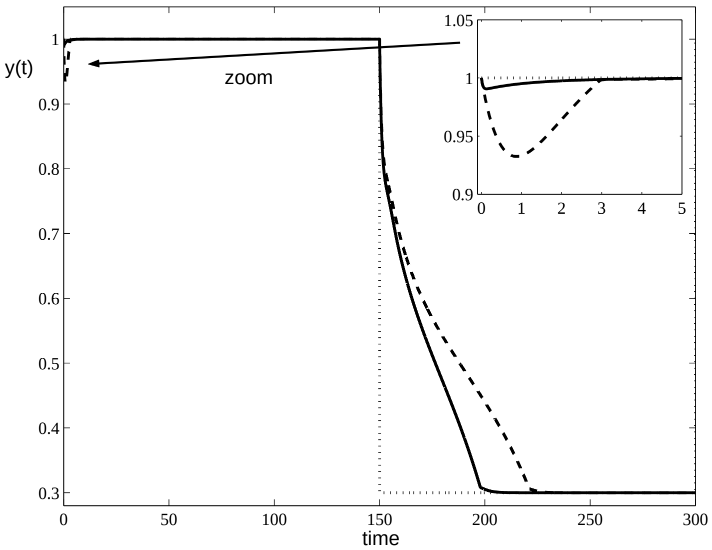

Figure 8: Reference signal (dotted), output response of algorithms TC (dashed) and 1DoF-TC (solid)

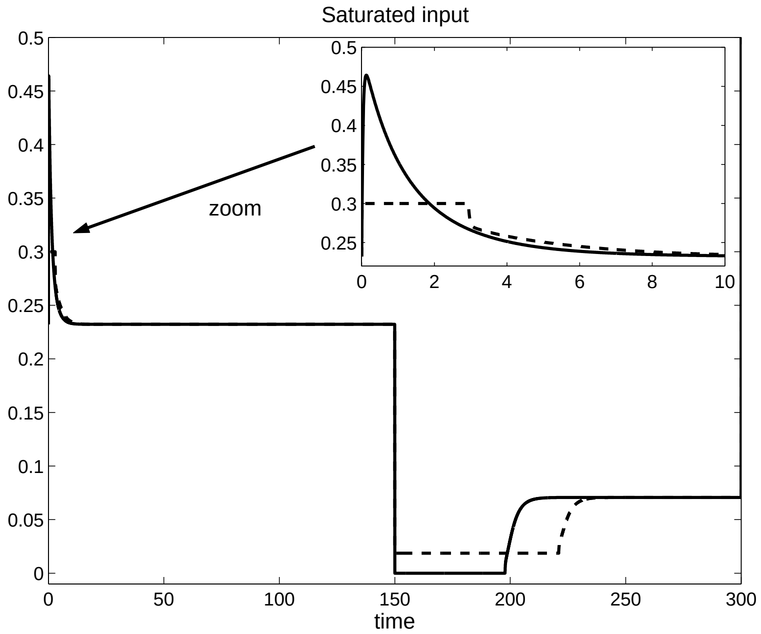

Figure 9: Saturated input of algorithms TC (dashed) and 1DoF-TC (solid).

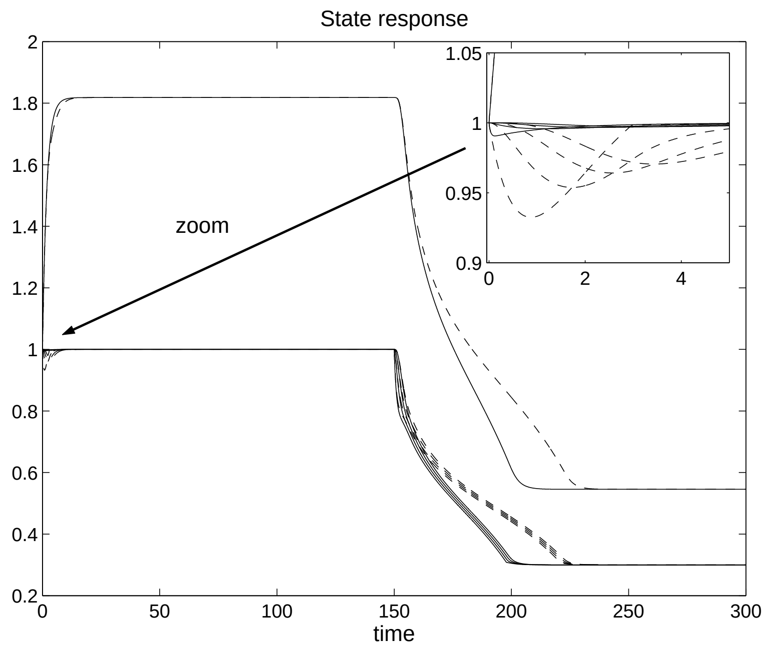

Figure 10: State response of algorithms TC (dashed) and 1DoF-TC (solid)

7 Conclusions

The paper has addressed tracking control of monotone nonlinear system in the presence of input and state constraints. It has been shown that for a certain class of nonlinear monotone systems, it is possible to design off-line a stabilizing static output controller in a straightforward way and then take into account on-line state and input constraints by saturation. The proposed controller operates, therefore, in two steps. In the first step, it computes state-dependent bounds on the input that guarantee state constraint feasibility at any future time instant. In the second step, it saturates the off-line designed control law among the on-line computed bounds thus implicitly acting as a reference governor. It has been proved that this jointly guarantees asymptotic tracking of a constant feasible setpoint and pointwise in time constraint fulfilment provided that an initial feasibility condition holds. Further, the implementation of the proposed control strategy is simple and exhibits a mild on-line computational burden. The effectiveness of the proposed procedure has been illustrated by means of two numerical examples of practical interest, concerning a linear and a nonlinear monotone system. Finally, an heuristic modification of the control strategy has been introduced for performance improvement. The stability properties of this modified control strategy are still not understood and will be the subject of future investigation.

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Appendix

Proof of Theorem 2 - In order to prove the theorem, the following lemma is fundamental.

Lemma 1 - If both systems (1) and (13) are monotone with respect to the same partial order, then the closed-loop system under the saturated feedback (14) is monotone.
Proof - By proposition 1 one needs to show the following for all $r \in R$

$$x_1 \succeq x_2 \Rightarrow f(x_1, \mathrm{sat}(\ell(h(x_1), r))) \succeq f(x_2, \mathrm{sat}(\ell(h(x_2), r))) \tag{46}$$

Only the following two cases need to be considered (the other are trivial)

1. $$\ell(h(x_1), r) \geq \mathrm{sat}(\ell(h(x_1), r)) \geq \mathrm{sat}(\ell(h(x_2), r)) \geq \ell(h(x_2), r) \tag{47}$$

2. $$\ell(h(x_1), r) \leq \mathrm{sat}(\ell(h(x_1), r)) \leq \mathrm{sat}(\ell(h(x_2), r)) \leq \ell(h(x_2), r) \tag{48}$$

In the first case, condition (46) is immediately verified by applying condition (3) to the open-loop system and letting $u_1 = \mathrm{sat}(\ell(h(x_1), r))$ and $u_2 = \mathrm{sat}(\ell(h(x_2), r))$. In the second case one has the following equality

$$\begin{aligned} &f(x_1, \mathrm{sat}(\ell(h(x_1), r))) - f(x_2, \mathrm{sat}(\ell(h(x_2), r))) = f(x_1, \mathrm{sat}(\ell(h(x_1), r)) - f(x_1, \ell(h(x_1), r)) + \\ &+ f(x_1, \ell(h(x_1), r)) - f(x_2, \ell(h(x_2), r)) + f(x_2, \ell(h(x_2), r)) - f(x_2, \mathrm{sat}(\ell(h(x_2), r))) \end{aligned} \tag{49}$$

Considering the relations in (48) and the monotonicity of the open-loop and closed-loop systems, it is straightforward to conclude that

$$\begin{aligned} &f(x_1, \mathrm{sat}(\ell(h(x_1), r)) \succeq f(x_1, \ell(h(x_1), r)) \\ &f(x_1, \ell(h(x_1), r)) \succeq f(x_2, \ell(h(x_2), r)) \\ &f(x_2, \ell(h(x_2), r)) \succeq f(x_2, \mathrm{sat}(\ell(h(x_2), r))) \end{aligned} \tag{50}$$

for all $x_1 - x_2 \in C$. Then condition (46) for monotonicity holds by convexity of the cones. ■

Proof of Theorem 2 - Since under the feedback $u = \ell(y, r)$ the closed-loop system (13) is monotone and has bounded trajectories then, from theorem 1, the unique equilibrium point $x_e(r) \in X$ is globally asymptotically stable in X. Lemma 1 asserts that also the closed-loop system under the saturated feedback (14) is monotone. Under the assumption that $x(t+1) = f(x(t), \underline{u})$ and $x(t+1) = f(x(t), \overline{u})$ have bounded trajectories in X, the system $x(t+1) = f(x(t), \mathrm{sat}(\ell(h(x(t)), r)))$ has bounded trajectories in X. Moreover the system has only an equilibrium point for all $r \in R$ and once again theorem 1 applies and $x(t+1) = f(x(t), \mathrm{sat}(\ell(h(x(t)), r)))$ is asymptotically stable in $x_e(r)$ for all $r \in R$ and globally in X. ■

Proof of Proposition 2 - The Jacobian of the system (13) is

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial \ell}{\partial h}\frac{\partial h}{\partial x} \tag{51}$$

It is immediate to see, evaluating (51) in (x, r) for each r and exploiting (1) and (2), that one gets the following equation structure

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\gamma e_i^T = \frac{\partial f}{\partial x} + \beta e_i \gamma e_i^T \tag{52}$$

where e_i is a vector with all 0s except for a 1 in the $i-th$ entry, $\beta = \frac{\partial f_i}{\partial u} \in \mathbb{R}$ and $\gamma = \frac{\partial \ell}{\partial h}\frac{\partial h}{\partial x_i} \in \mathbb{R}$. This means that the term $\beta e_i \gamma e_i^T$ does not affect the off-diagonal elements of $\frac{\partial f}{\partial x}$ and, hence, the monotonicity of the system is preserved. ■

Proof of Theorem 3 - From the definition of monotone systems for any $x \in S$ the following relation of order is met

$$u_1(\cdot) > u_2(\cdot) \Rightarrow \varphi(t, x, u_1(\cdot)) \succeq \varphi(t, x, u_2(\cdot)) \quad \forall t \geq 0 \tag{53}$$

Then, considering relation (53) under the assumption of monotone constraints, there exist constant $\underline{u}_x$ and $\overline{u}_x$, with $\overline{u}_x > \underline{u}_x$, given by (16), such that

$$\overline{g}_i \geq g_i(\varphi(t, x, \overline{u}_x 1(\cdot))) \geq g_i(\varphi(t, x, \underline{u}_x 1(\cdot))) \geq \underline{g}_i, \quad \forall i \text{ and } \forall\ t \geq 0 \tag{54}$$

Since for any signal $u(\cdot)$ such that $\underline{u}_x \leq u(\cdot) \leq \overline{u}_x$

$$g_i(\varphi(t, x, \overline{u}_x 1(\cdot))) \geq g_i(\varphi(t, x, u(\cdot))) \geq g_i(\varphi(t, x, \underline{u}_x 1(\cdot))) \tag{55}$$

the state constraints $x \in S$ can be replaced with $\underline{u}_x \leq u(\cdot) \leq \overline{u}_x$ ■

Proof of Theorem 4 - Notice that it is possible to restrict our attention to cooperative systems by an appropriate change of coordinates [7]. A system is excitable if any state variable x_i can be influenced directly or indirectly by the input u. Then, in terms of the incidence graph, there must exist a directed path between u and each node x_i. Obviously the shortest path will be not greater than n since it has been assumed, by (20) and (21), that the interactions between vertices are strictly monotone. Conversely we show by induction that if x_i is reachable from u through a directed path of length ℓ in the incidence graph then, for any $u_1(\cdot) > u_2(\cdot)$ and for any $x \in X$, $\varphi_i(\ell, x, u_1(\cdot)) > \varphi_i(\ell, x, u_2(\cdot))$. The claim is trivial for $\ell = 1$ (by definition of the incidence graph). Assume that the claim is true for $\ell - 1$ and let x_j, $j \neq i$, be the $(\ell - 1)$-th node along the path from u to x_i. Then, by induction, $\varphi_j(\ell - 1, x, u_1(\cdot)) > \varphi_j(\ell - 1, x, u_2(\cdot))$ which, in turn, implies $f_i(\varphi(\ell - 1, x, u_1(\cdot)), u_1(\ell - 1)) > f_i(\varphi(\ell - 1, x, u_2(\cdot)), u_2(\ell - 1))$, i.e. $\varphi_i(\ell, x, u_1(\cdot)) > \varphi_i(\ell, x, u_2(\cdot))$. This proves the claim. ■

Proof of Theorem 5 - Since $[\max\{\underline{u}, \underline{u}_{x(0)}\},\ \min\{\overline{u}, \overline{u}_{x(0)}\}] \neq \emptyset$, given the state $x(0) = x_0$, the constraints $\varphi(t, x_0, u(\cdot)) \in S$ are satisfied for all $t \geq 0$ with $\underline{u}_{x_0} \leq u(\cdot) \leq \overline{u}_{x_0}$ by virtue of theorem 3. Let us consider the situation in which $\underline{u}_S < \underline{u}_{x_0} \leq \overline{u}_{x_0} < \overline{u}_S$. In fact if $\overline{u}_{x_0} = \overline{u}_S$ and $\underline{u}_{x_0} = \underline{u}_S$ the result follows directly from theorem 2.
At each time instant t the tracking control algorithm computes new bounds for the input and by virtue of theorem 3 the following monotonicity condition is satisfied $\underline{u}_{x(t)} \leq \underline{u}_{x(t')} \leq \overline{u}_{x(t')} \leq \overline{u}_{x(t)}$ for $0 < t' < t$, so that the optimization problem is always feasible and constraints are always

satisfied along the system trajectories. We need to show convergence of $x(t)$ to $x_e(r_d)$. By monotonicity $\underline{u}_\infty = \lim_{t\to\infty} \underline{u}_{x(t)}$ and $\overline{u}_\infty = \lim_{t\to\infty} \overline{u}_{x(t)}$ are well defined. We claim that $\overline{u}_S = \overline{u}_\infty$ and similarly $\underline{u}_S = \underline{u}_\infty$.

By contradiction assume that $\overline{u}_\infty < \overline{u}_S$. Due to the monotonicity of the system, $\varphi(t, x_0, u(\cdot)) \preceq \varphi(t, x_0, \overline{u}_S\ 1(\cdot))$ for any $u(\cdot)$ satisfying $\underline{u}_{x(t)} \leq u(\cdot) \leq \overline{u}_{x(t)}$ for all $t > 0$.

Under the assumption of excitability of the system there exists $\tilde{t}$ such that for all $t > \tilde{t}$

$$\varphi(t, x_0, u(\cdot)) \preceq \varphi(t, x_0, \overline{u}_\infty\ 1(\cdot)) \ll \varphi(t, x_0, \overline{u}_S\ 1(\cdot))$$

Let us consider the $\omega-$limit set of x_0 with input $\overline{u}_\infty\ 1(\cdot)$

$$\Omega(x_0,\ \overline{u}_\infty\ 1(\cdot)) \stackrel{\triangle}{=} \left\{ x\ :\ \exists\ t_n \longrightarrow \infty\ :\ \lim_{n\to\infty} \varphi(t_n, x_0, \overline{u}_\infty\ 1(\cdot)) = x \right\}$$

Obviously, by monotonicity of the system, $\Omega(x_0,\ \overline{u}_\infty\ 1(\cdot)) \preceq x_{\overline{u}_S}$ where $x_{\overline{u}_S}$ is the unique equilibrium point corresponding to $\overline{u}_S$. Let us consider an arbitrary $x \in \Omega(x_0,\ \overline{u}_\infty\ 1(\cdot))$ and the corresponding trajectories with initial conditions $x \preceq x_{\overline{u}_S}$ and inputs $\overline{u}_\infty\ 1(\cdot) \prec \overline{u}_S$. For the condition of uniform excitability of the monotone system and the uniqueness of the equilibrium point $x_{\overline{u}_S}$, the following relation holds

$$\varphi(t, x, \overline{u}_\infty\ 1(\cdot)) \ll \varphi(t, x_{\overline{u}_S}, \overline{u}_S\ 1(\cdot)) = x_{\overline{u}_S}, \quad \forall t > \tilde{t}.$$

Since $\varphi(t, x, \overline{u}_\infty\ 1(\cdot)) \in \Omega(x_0,\ \overline{u}_\infty\ 1(\cdot))$ for all $t \geq 0$ and x is an arbitrary point in $\Omega(x_0,\ \overline{u}_\infty\ 1(\cdot))$, it turns out that

$$\Omega(x_0,\ \overline{u}_\infty\ 1(\cdot)) \ll x_{\overline{u}_S}.$$

This means that there exists $\check{t}$ such that $\varphi(t, x_0, \overline{u}_\infty\ 1(\cdot)) \preceq x_{\overline{u}_S}$ for all $t > \check{t}$ and considering the k-steps-ahead state prediction, computed applying a constant input $\overline{u}_S\ 1(\cdot)$, we have

$$\varphi(k, \varphi(t, x_0, u(\cdot)), \overline{u}_S\ 1(\cdot)) \preceq \varphi(k, \varphi(t, x_0, \overline{u}_\infty\ 1(\cdot)), \overline{u}_S\ 1(\cdot)) \preceq \varphi(k, x_{\overline{u}_S}, \overline{u}_S\ 1(\cdot))$$

Hence, for all $k \geq 0$, $\varphi(k, \varphi(t, x_0, \overline{u}_\infty\ 1(\cdot)), \overline{u}_S\ 1(\cdot)) \in S$. This implies $\overline{u}_\infty \geq \overline{u}_S$ for all $t > \check{t}$ and contradicts the assumption $\overline{u}_\infty < \overline{u}_S$. Similarly $\exists\ \hat{t} > 0$ such that $\underline{u}_\infty \leq \underline{u}_S$ is satisfied for all $t \geq \hat{t}$. If convergence in finite time of $\overline{u}_{x(t)}$ to $\overline{u}_S$ could be shown, then theorem 2 would apply and give global asymptotic stability. Since convergence is, however, only asymptotic, our system may be interpreted as a time-varying system (with a saturation of growing amplitude), and then one can resort to the theory of asymptotically autonomous systems. It is well known that, if the limiting system (viz. the one with a constant saturation range equal to $[\underline{u}_S,\ \overline{u}_S]$) is globally asymptotically stable, the time-varying system (viz. the one with a time-varying saturation range equal to $[\underline{u}_{x(t)},\ \overline{u}_{x(t)}]$) is also globally asymptotically stable provided that solutions are bounded (in [20] this result was proved for continuous-time systems, but the same argument can be applied to discrete time systems as well). This concludes the proof of the theorem. ■

Proof of Theorem 6 - Since the system (25) is asymptotically stable, given any compact set $K \subset \mathbb{R}^n$, it holds that

$$\forall \delta > 0,\ \exists\ \bar{t}_\delta \mid \forall t \geq \bar{t}_\delta, \forall x \in K:\ \|A^t x\|_\infty \leq \delta.$$

In order to obtain a $\overline{t}_\delta$ that is valid for any $x_0 \in S$ and for any $u \in U_{S_\delta}$, due to the system's monotonicity, it is sufficient to compute such $\overline{t}_\delta$ for the initial conditions $\underline{x} = \inf_{x \in S,\ u \in U_{S_\delta}} (x - H(1)u)$ and $\overline{x} = \sup_{x \in S,\ u \in U_{S_\delta}} (x - H(1)u)$ with respect to the partial order in X, where $H(1) = (I - A)^{-1}B$ and I denotes the identity matrix. More precisely, let

$$\overline{t}_\delta = \inf\{t : \|A^k \underline{x}\|_\infty \leq \delta \text{ and } \|A^k \overline{x}\|_\infty \leq \delta, \forall k \geq t\}.$$

Let $x_0 \in S$ be arbitrary and $u \in U_{S_\delta}$. By linearity, $\varphi(t, x_0, u\ 1(\cdot)) = A^t(x_0 - H(1)u) + H(1)u$. Then, by monotonicity,

$$A^t \underline{x} + H(1)u \leq \varphi(t, x_0, u\ 1(\cdot)) \leq A^t \overline{x} + H(1)u.$$

Since, for all $t \geq \overline{t}_\delta$, $H(1)u \in S_\delta$, $\|A^t \overline{x}\|_\infty \leq \delta$ and $\|A^t \underline{x}\|_\infty \leq \delta$, then there exist $s_1, s_2 \in S$ such that

$$s_1 \preceq \varphi(t, x_0, u\ 1(\cdot)) \preceq s_2$$

Since S is described by monotone constraints then also $\varphi(t, x_0, u\ 1(\cdot)) \in S$. This proves the existence of $\overline{t} = \overline{t}_\delta$ in (26). ■