Counting derangements with ascents and descents in given positions

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Abstract

A derangement is a permutation without fixed points. There are several generalizations of derangements in the literature. Eriksen, Freij and Wästlund [2] recently have studied derangements with descents in given positions and ask what can be said for derangements with ascents instead of descents in given positions. Counting derangements with ascents and descents in given positions is studied.

1 Derangements

Definition. A derangement is a permntation without fixed points, i.e. $\pi = \pi_1\pi_2\cdots\pi_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$ such that $\pi_i \neq i$ for all $i = 1, 2, \ldots, n$.

Equivalently, a derangement is a permutation without 1-cycles.

$$3\,4\,5\,2\,1\,8\,6\,7 = \left(\frac{1\,2\,3\,4\,5\,6\,7\,8}{3\,4\,5\,2\,1\,8\,6\,7}\right) = (1\,3\,5)(2\,4)(6\,8\,7)$$

2 Colored permutations

Colored permutation: A pemutation of colored elements.

Let π be a permutation of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of type $\mathbf{a} = (a_1, a_2, \ldots, a_k) = (3, 3, 3)$ with three color blocks of sizes 3, 3 and 3. Define various generalized derangements:

- (NFCo) No fixed colors: $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,9\,6\,8\,1\,7\,3\,4\,2 \end{pmatrix} = (1\,5)(2\,9)(3\,6\,7)(4\,8)$
- (NFiD) No fixed points when elements are rearranged in descending order in each color block: $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,2\,7\,1\,8\,4\,3\,6\,9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,2\,3 \mid 4\,5\,6 \mid 7\,8\,9 \\ \hline 7\,5\,2 \mid 8\,4\,1 \mid 9\,6\,3 \end{pmatrix}$
- (NFiA) No fixed points when elements are rearranged in ascending order in each color block: $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,2\,7\,1\,9\,6\,3\,8\,4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,2\,3 \mid 4\,5\,6 \mid 7\,8\,9 \\ \hline 2\,5\,7 \mid 1\,6\,9 \mid 3\,4\,8 \end{pmatrix}$
- (NMCy) No monochromatic cycles: $(1\,2\,4\,8)(3\,6)(5\,7\,9)$

- (NFiM) No fixed points when elements are rearranged in mixed (ascending or descending) order in each color block:

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,2\,7\,1\,9\,6\,3\,8\,4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8\,9 \\ \hline 2\,5\,7\,|\,9\,6\,1\,|\,3\,4\,8 \end{pmatrix}$$

Note that if each block is of size 1 then all five reduce to the usual derangements.

3 NFCo: No fixed colors

For $\mathbf{a} = (3, 3, 3)$,

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,9\,6\,8\,1\,7\,3\,4\,2 \end{pmatrix} = (1\;5)(2\;9)(3\;6\;7)(4\;8).$$

This version of generalized derangements is associated to combinatorics for Laguerre polynomials. The linearization coefficients of Laguerre polynomials is the generating function of NFCo-derangements. Many people have published in this version, including Gillis, Even, Foata, Zeilberger, ...

4 NFiD: No fixed points in descending order

For $\mathbf{a} = (3, 3, 3)$,

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,2\,7\,1\,8\,4\,3\,6\,9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8\,9 \\ \hline 7\,5\,2\,|\,8\,4\,1\,|\,9\,6\,3 \end{pmatrix}.$$

For a composition $\mathbf{a}$ of n, let $D^{\mathrm{NFiD}}(\mathbf{a})$ denote the set of all NFiD-derangements in S_n.

5 NFiD: No fixed points in descending order

Example. For $n = 6$, $\mathbf{a} = (4, 2)$ and the underlying set $\{1, 2, 3, 4, 5, 6\}$, after rearranging each block in descending order:

$$6543|21 = \begin{pmatrix} 1\,2\,3\,4\,|\,5\,6 \\ \hline 6\,5\,4\,3\,|\,2\,1 \end{pmatrix}$$

$$6542|31 \qquad 6541|32 \qquad 6521|43$$

$$5421|63 \qquad 5321|64 \qquad 4321|65$$

So, $|D^{\mathrm{NFiD}}(4, 2)| = 7 \times 4! \times 2!$.

This version, with a slightly different definition, has been considered by Guo-Niu Han and Guoce Xin [1], Niklas Eriksen, Ragnar Freij, Johan Wästlund [2]. Some questions are raised in [2], one of which is *What will happen if the ascending order is used?*

Theorem 1. *(Eriksen, Freij and Wästlund [2, Theorem 2.1])*

$$\sum_{\mathbf{a}} D^{\mathrm{NFiD}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} = \frac{1}{(1 - \sum_{i=1}^{k} x_i) \times \prod_{i=1}^{k} (1 + x_i)},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, $\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^{k} x_i^{a_i}$ *and* $\mathbf{a}! = \prod_{i=1}^{k} a_i!$.

6 NFiA: No fixed points in ascending order

For $\mathbf{a} = (3, 3, 3)$,

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,2\,7\,1\,9\,6\,3\,8\,4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8\,9 \\ \hline 2\,5\,7\,|\,1\,6\,9\,|\,3\,4\,8 \end{pmatrix}.$$

For a composition $\mathbf{a}$ of n, let $D^{\mathrm{NFiA}}(\mathbf{a})$ denote the set of all NFiA-derangements in S_n.

7 NFiA: No fixed points in ascending order

Example. For $n = 6$, $\mathbf{a} = (4, 2)$ and the underlying set $\{1, 2, 3, 4, 5, 6\}$, after rearranging each block in ascending order:

$$3456|12 = \begin{pmatrix} 1\,2\,3\,4\,|\,5\,6 \\ \hline 3\,4\,5\,6\,|\,1\,2 \end{pmatrix}$$

$$2456|13 \qquad 2356|14 \qquad 2346|15$$

So, $|D^{\text{NFiA}}(4, 2)| = 4 \times 4! \times 2! = 192$.

Theorem 2 (Kim and Seo [3])**.**

$$\sum_{\mathbf{a}} D^{\text{NFiA}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} = \frac{\prod_{i=1}^{k}(1 - x_i)}{1 - \sum_{i=1}^{k} x_i},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, $\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^{k} x_i^{a_i}$ *and* $\mathbf{a}! = \prod_{i=1}^{k} a_i!$.

8 NMCy: No monochromatic cycles

For $\mathbf{a} = (3,3,3)$,

$$(1\,2\,4\,8)(3\,6)(5\,7\,9) = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 2\,4\,6\,8\,7\,3\,9\,1\,5 \end{pmatrix}.$$

For a composition $\mathbf{a}$ of n, let $D^{\mathrm{NMCy}}(\mathbf{a})$ denote the set of all NMCy-derangements in S_n.

Example. For $n = 6$, $\mathbf{a} = (4,2)$ and the underlying set $\{1,2,3,4,5,6\}$, there are three different types:

- rb-rrrb: two mixed cycles of length 2 and 4 $\rightarrow 4 \times 2 \times 3! = 48$
- rrb-rrb: two mixed cycles of length 3 and 3 $\rightarrow 6 \times 2! \times 2! = 24$
- rrrrbb: one mixed cycle of length 6 $\rightarrow 5! = 120$

  So, $|D^{\mathrm{NMCy}}(4,2)| = 48 + 24 + 120 = 192$

Theorem 3 (Kim and Seo [3])**.**

$$\sum_{\mathbf{a}} D^{\mathrm{NMCy}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} = \frac{\prod_{i=1}^{k}(1-x_i)}{1-\sum_{i=1}^{k} x_i},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, $\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^{k} x_i^{a_i}$ *and* $\mathbf{a}! = \prod_{i=1}^{k} a_i!$.

9 Generating functions of NFiA and NMCy

$$\frac{\prod_{i=1}^{k}(1-x_i)}{1-\sum_{i=1}^{k} x_i} = \sum_{\mathbf{a}} D^{\mathrm{NFiA}}(\mathbf{a})\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

$$= \sum_{\mathbf{a}} D^{\mathrm{NMCy}}(\mathbf{a})\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

10 A combinatorial bijection from NFiA to NMCy

Recall the involution principle by Garsia and Milne.

- A nonempty set U with a weight fuction $w : U \to R$. (R is a ring.)
- Two weight-preserving sign-reversing involutions Φ and Ψ on U with fixed sets F_Φ and F_Ψ respectively.

$$w(\Phi(u)) = -w(u), w(\Psi(u)) = -w(u) \text{ for all } u \in U.$$

- Then

$$\sum_{u \in U} w(u) = \sum_{u \in F_\Phi} w(u) = \sum_{u \in F_\Psi} w(u)$$

- There is a bijection from F_Φ to F_Ψ by applying $\Phi \circ \Psi \circ \cdots \circ \Phi \circ \Psi(u)$

$$u \text{—} \Psi(u) \text{—} \Phi \circ \Psi(u) \text{—} \cdots \text{—} (\Phi \circ \Psi)^l(u) = v$$

with $u = \Phi(u)$ and $v = \Psi(v)$.

11 A combinatorial involution Φ for NFiA

Fix k, the number of colors.

For a given permutation π, $a_i(\pi)$ denotes the number of integers of color i in π.

- Universe: The set of all $(\pi; s_1, s_2, \ldots, s_k)$'s such that for each i, $s_i = \emptyset$ or an integer in $\{1, 2, \ldots, a_i(\pi) + 1\}$.
- Weight: $(-1)^{\#\text{ of integers among } s_i\text{'s}} \times \frac{\mathbf{x}^*}{\mathbf{a}^*!}$
- If s_i is an integer, it is movable.
- If s_i is the empty set and there are fixed points in a color block, the largest among them is movable.
- Move the largest movable element.

Details by examples.

12 A combinatorial involution Φ for NFiA

$$\frac{1}{1-\sum_{i=1}^{k} x_i} \times \prod_{i=1}^{k}(1-x_i) = \sum_{\mathbf{a}} D^{\mathrm{NFiA}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

$$\pi = \left(\frac{1\,2\,3\,4\,5\,6\,7\,8}{4\,1\,8\,5\,\underline{6}\,3\,7\,2}\right) \to \left(\frac{1\,2\,3\,|\,4\,5\,6\,|\,7\,8}{1\,4\,8\,|\,3\,5\,6\,|\,2\,7}\right)$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\emptyset}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \underline{2}, \emptyset)$$

$$\pi' = \left(\frac{1\,2\,3\,4\,5\,6\,7}{4\,1\,7\,5\,3\,6\,2}\right) \to \left(\frac{1\,2\,3\,|\,4\,5\,|\,6\,7}{1\,4\,7\,|\,3\,5\,|\,2\,6}\right)$$

13 A combinatorial involution Φ for NFiA

$$\pi = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ \hline 4\,1\,8\,5\,6\,3\,7\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8 \\ \hline 1\,4\,8\,|\,3\,5\,6\,|\,2\,7 \end{pmatrix}$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{1}}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 4\,1\,9\,\underline{7}\,5\,6\,3\,8\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,7\,|\,8\,9 \\ \hline 1\,4\,9\,|\,3\,5\,6\,7\,|\,2\,8 \end{pmatrix}$$

14 A combinatorial involution Φ for NFiA

$$\pi = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ \hline 4\,1\,8\,5\,6\,3\,7\,2 \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8 \\ \hline 1\,4\,8\,|\,3\,5\,6\,|\,2\,7 \end{pmatrix}$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{2}}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 4\,1\,9\,5\,\underline{7}\,6\,3\,8\,2 \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,7\,|\,8\,9 \\ \hline 1\,4\,9\,|\,3\,5\,6\,7\,|\,2\,8 \end{pmatrix}$$

15 A combinatorial involution Φ for NFiA

$$\pi = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ \hline 4\,1\,8\,5\,6\,3\,7\,2 \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8 \\ \hline 1\,4\,8\,|\,3\,5\,6\,|\,2\,7 \end{pmatrix}$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{3}}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 4\,1\,9\,5\,6\,\underline{7}\,3\,8\,2 \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,7\,|\,8\,9 \\ \hline 1\,4\,9\,|\,3\,5\,6\,7\,|\,2\,8 \end{pmatrix}$$

16 A combinatorial involution Φ for NFiA

$$\pi = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ \hline 4\,1\,8\,5\,6\,3\,7\,2 \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8 \\ \hline 1\,4\,8\,|\,3\,5\,6\,|\,2\,7 \end{pmatrix}$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{4}}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 4\,1\,9\,5\,6\,3\,\underline{7}\,8\,2 \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,7\,|\,8\,9 \\ \hline 1\,4\,9\,|\,3\,5\,6\,7\,|\,2\,8 \end{pmatrix}$$

17 What are the fixed points of Φ?

$$\frac{\prod_{i=1}^{k}(1-x_i)}{1-\sum_{i=1}^{k} x_i} = \sum_{\mathbf{a}} D^{\mathrm{NFiA}}(\mathbf{a})\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

Answer: Exactly $(\pi;\emptyset,\ldots,\emptyset)$ where π is an NFiA-derangement.

18 A combinatorial involution Ψ for NMCy

Fix k, the number of colors.

For a given permutation π, $a_i(\pi)$ denotes the number of integers of color i in π.

- Universe: The set of all $(\pi; s_1, s_2, \ldots, s_k)$'s such that for each i, $s_i = \emptyset$ or an integer in $\{1, 2, \ldots, a_i(\pi) + 1\}$.
- Weight: $(-1)^{\#\text{ of integers among } s_i\text{'s}} \times \frac{\mathbf{x}^*}{\mathbf{a}^*!}$
- If s_i is an integer, it is movable.
- If s_i is the empty set and there are some monochromatic cycles of color i, the preimage of the largest elements among them is movable.
- Move the largest movable element.

Details by examples.

19 A combinatorial involution Ψ for NMCy

$$\frac{1}{1-\sum_{i=1}^{k} x_i} \times \prod_{i=1}^{k}(1-x_i) = \sum_{\mathbf{a}} D^{\mathrm{NMCy}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

$$\pi = (1\,2\,7\,8)(3\,5)(\underline{4}\,6) = \begin{pmatrix} 1\,2\,3 \,|\, 4\,5\,6 \,|\, 7\,8 \\ \hline 2\,7\,5 \,|\, 6\,3\,4 \,|\, 8\,1 \end{pmatrix}$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\emptyset}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \underline{1}, \emptyset)$$

$$\pi' = (1\,2\,6\,7)(3\,4)(5) = \begin{pmatrix} 1\,2\,3 \,|\, 4\,5 \,|\, 6\,7 \\ \hline 2\,6\,4 \,|\, 3\,5 \,|\, 7\,1 \end{pmatrix}$$

20 A combinatorial involution Ψ for NMCy

$$\pi = (1\,2\,7\,8)(3\,5)(4\,6) = \left(\frac{1\,2\,3\,|\,4\,5\,6\,|\,7\,8}{2\,7\,5\,|\,6\,3\,4\,|\,8\,1}\right)$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{1}}, \emptyset)$$

$$(\pi'; s'_1, s'_2, s'_3) = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = (1\,2\,8\,9)(3\,6)(5\,\underline{4}\,7) = \left(\frac{1\,2\,3\,|\,4\,5\,6\,7\,|\,8\,9}{2\,8\,6\,|\,7\,4\,3\,5\,|\,9\,1}\right)$$

21 A combinatorial involution Ψ for NMCy

$$\pi = (1\,2\,7\,8)(3\,5)(4\,6) = \left(\frac{1\,2\,3\,|\,4\,5\,6\,|\,7\,8}{2\,7\,5\,|\,6\,3\,4\,|\,8\,1}\right)$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{2}}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = (1\,2\,8\,9)(3\,6)(4\,\underline{5}\,7) = \left(\frac{1\,2\,3\,|\,4\,5\,6\,7\,|\,8\,9}{2\,8\,6\,|\,5\,7\,3\,4\,|\,9\,1}\right)$$

22 A combinatorial involution Ψ for NMCy

$$\pi = (1\,2\,7\,8)(3\,5)(4\,6) = \left(\frac{1\,2\,3\,|\,4\,5\,6\,|\,7\,8}{2\,7\,5\,|\,6\,3\,4\,|\,8\,1}\right)$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{3}}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = (1\,2\,8\,9)(3\,5)(4\,\underline{6}\,7) = \left(\frac{1\,2\,3\,|\,4\,5\,6\,7\,|\,8\,9}{2\,8\,5\,|\,6\,3\,7\,4\,|\,9\,1}\right)$$

23 A combinatorial involution Ψ for NMCy

$$\pi = (1\,2\,7\,8)(3\,5)(4\,6) = \left(\frac{1\,2\,3 \mid 4\,5\,6 \mid 7\,8}{2\,7\,5 \mid 6\,3\,4 \mid 8\,1}\right)$$

$$(\pi; s_1, s_2, s_3) = (\pi; 2, \boxed{\underline{4}}, \emptyset)$$

$$(\pi'; s_1', s_2', s_3') = (\pi; 2, \emptyset, \emptyset)$$

$$\pi' = (1\,2\,8\,9)(3\,5)(4\,6)(\underline{7}) = \left(\frac{1\,2\,3 \mid 4\,5\,6\,7 \mid 8\,9}{2\,8\,5 \mid 6\,3\,4\,7 \mid 9\,1}\right)$$

24 What are the fixed points of Ψ?

$$\frac{\prod_{i=1}^{k}(1-x_i)}{1-\sum_{i=1}^{k}x_i} = \sum_{\mathbf{a}} D^{\mathrm{NMCy}}(\mathbf{a})\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

Answer: Exactly $(\pi; \emptyset, \ldots, \emptyset)$ where π is an NMCy-derangement.

25 Combinatorial bijections for NFiA and NMCy?

$$\sum_{\mathbf{a}} D^{\mathrm{NFiA}}(\mathbf{a})\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} = \frac{\prod_{i=1}^{k}(1-x_i)}{1-\sum_{i=1}^{k}x_i} = \sum_{\mathbf{a}} D^{\mathrm{NMCy}}(\mathbf{a})\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

- These identities are proved by weight-preserving-sign-reversing involutions whose fixed sets consist of positive elements only.
- By the involution principle of Garsia and Milne, there is a combinatorial bijection between two fixed sets.
- Explicit description of the bijection seems to be hard to find.
- Is there a simple combinatorial bijection between NFiA-derangements and NMCy-derangements?

26 NFiM: No fixed points in mixed (ascending or descending) order

For $\mathbf{a} = (3, 3, 3)$, with ascending|descending|ascending order,

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 5\,2\,7\,1\,9\,6\,3\,8\,4 \end{pmatrix} \to \begin{pmatrix} 1\,2\,3\,|\,4\,5\,6\,|\,7\,8\,9 \\ \hline 2\,5\,7\,|\,9\,6\,1\,|\,3\,4\,8 \end{pmatrix}.$$

For a composition $\mathbf{a}$ of n, let $D^{\text{NFiM}}(\mathbf{a})$ denote the set of all NFiM-derangements in S_n.

27 NFiM: No fixed points in mixed order

Example. For $n = 6$, $\mathbf{a} = (4, 2)$ and the underlying set $\{1, 2, 3, 4, 5, 6\}$, after rearranging the first block in descending and the second in ascending order:

$$6543|12 = \begin{pmatrix} 1\,2\,3\,4\,|\,5\,6 \\ \hline 6\,5\,4\,3\,|\,1\,2 \end{pmatrix}$$

$$6542|13 \qquad 6541|23 \qquad 6521|34$$

$$6421|35 \qquad 6321|45$$

So, $|D^{\text{NFiM}}(4, 2)| = 6 \times 4! \times 2! = 288.$

28 More on NFiM-derangements

Theorem 4 (Kim and Seo [3]). *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ *be a composition of an nonnegative integer and* H *a subset of* $[k] = \{1, 2, \ldots, k\}$.

In $D^{\mathrm{NFiM}}(\mathbf{a})$, *elements are ordered in ascending order in each block* i *in* H *and in descending order in other blocks. Then*

$$\sum_{\mathbf{a}} D^{\mathrm{NFiM}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} = \frac{\prod_{i \in H}(1 - x_i)}{(1 - \sum_{i=1}^{k} x_i) \times \prod_{i \in [k] \setminus H}(1 + x_i)},$$

where $\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^{k} x_i^{a_i}$ *and* $\mathbf{a}! = \prod_{i=1}^{k} a_i!$.

29 Cycle index of S_n

For a permutation π with $c_i(\pi)$ i-cycles, let

$$\rho_\pi(\mathbf{z}) = \prod_{i=1}^{\infty} z_i^{c_i(\pi)}.$$

$$I_n(\mathbf{z}) = \sum_{\pi \in S_n} \rho_\pi(\mathbf{z}) = \sum_{\pi \in S_n} z_1^{c_1(\pi)} \cdots z_n^{c_n(\pi)}$$

$$\sum_{n \geq 0} I_n(\mathbf{z}) \frac{t^n}{n!} = \exp \sum_{i>0} z_i \frac{t^i}{i}$$

30 Cycle generating function for NMCy

Theorem 5. *The identity*

$$\sum_{\pi \in D^{\mathrm{NMCy}}(\mathbf{a})} \rho_\pi(\mathbf{z}) = \sum_{0 \le \mathbf{b} \le \mathbf{a}} I_{n-\|\mathbf{b}\|}(\mathbf{z}) \prod_i \binom{a_i}{b_i} I_{b_i}(-\mathbf{z})$$

holds.

The RHS of the identity suggests a natural weight-preserving sign-reversing involution whose fixed set consists of permutations without monochromatic cycles.

If we are just counting,

$$\begin{aligned}\left|D^{\mathrm{NMCy}}(\mathbf{a})\right| &= \sum_{\pi\in D^{\mathrm{NMCy}}(\mathbf{a})} \rho_\pi(1,1,\dots)\\ &= \sum_{0\le\mathbf{b}\le\mathbf{1}} (n-\|\mathbf{b}\|)!\prod_i \binom{a_i}{b_i}(-1)^{b_i}\\ &= \sum_{\alpha\subset[k]} (-1)^{|\alpha|}(n-|\alpha|)!\prod_{i\in\alpha} a_i!.\end{aligned}$$

For $\mathbf{a}=(4,2)$,

$$7!-5!\cdot 4-5!\cdot 2+4!\cdot 4\cdot 2=192.$$

31 (NFiA, NMCy) : (NFiD, ????)

Recall NMCy (No monochromatic cycles). For $\mathbf{a} = (3, 3, 3)$,

$$(1\,2\,4\,8)(3\,6)(5\,7\,9) = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 2\,4\,6\,8\,7\,3\,9\,1\,5 \end{pmatrix}.$$

For a composition $\mathbf{a}$ of n, let $D^{\text{NMCy}}(\mathbf{a})$ denote the set of all NMCy-derangements in S_n.

$$\sum_{\mathbf{a}} D^{\text{NFiA}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} = \frac{\prod_{i=1}^{k}(1-x_i)}{1-\sum_{i=1}^{k} x_i} = \sum_{\mathbf{a}} D^{\text{NMCy}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!}$$

32 NMCy1-derangements

- For any color the total number of elements in monochromatic cycles of the color must be even.
- Since 0 is even, any NMCy-derangement is also an NMCy1-derangement.
- Let $D^{\mathrm{NMCy1}}(\mathbf{a})$ denote the set of all NMCy1-derangements in S_n.

Theorem 6.

$$\sum_{\mathbf{a}} D^{\mathrm{NMCy1}}(\mathbf{a}) \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} = \frac{1}{(1 - \sum_{i=1}^{k} x_i) \times \prod_{i=1}^{k} (1 + x_i)},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, $\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^{k} x_i^{a_i}$ *and* $\mathbf{a}! = \prod_{i=1}^{k} a_i!$.

Theorem 7. *The identity*

$$\sum_{\pi \in D^{\mathrm{NMCy1v}}(\mathbf{a})} \rho_\pi(\mathbf{z}) = \sum_{0 \le \mathbf{b} \le \mathbf{a}} (-1)^{\|\mathbf{b}\|} I_{n-\|\mathbf{b}\|}(\mathbf{z}) \prod_i \binom{a_i}{b_i} I_{b_i}(\mathbf{z})$$

holds.

Note that NMCy1 and NMCy1v are slightly different.

The RHS of the identity suggests a natural weight-preserving sign-reversing involution whose fixed set(NMCy1v-derangements) consists of permutations without monochromatic cycles of odd length and with two kinds of monochromatic cycles of even length.

If we generate permutations with red and blue cycles of even lengths, we can generate all permutations of even lengths.

Since there is no 1-cycles in permutations in the fixed set, the above identity induces [2, Theorem 5.1].

$$\begin{aligned}\left|D^{\mathrm{NMCy1}}(\mathbf{a})\right| &= \sum_{\pi\in D^{\mathrm{NMCy1}}(\mathbf{a})} c_\pi(\lambda,1,\dots)\\ &= \sum_{0\le\mathbf{b}\le\mathbf{a}} (-1)^{\|\mathbf{b}\|} I_{n-\|\mathbf{b}\|}(\lambda,1,\dots)\prod_i \binom{a_i}{b_i} I_{b_i}(\lambda,1,\dots)\end{aligned}$$

33 (NFiM, NMCy2)

In NMCy2-derangements, monochromatic cycles are allowed in colors associated to descents. Recall the definition of NFiM-derangements.

Theorem 8. *Let H be a subset of $[k]$ and $\sigma = (\sigma_1, \ldots, \sigma_k)$, where*

$$\sigma_i = -1 \text{ if } i \in H;\ 1 \text{ otherwise.}$$

For any $\lambda \in \mathbb{C}$, the identity

$$\sum_{\pi \in D^{\mathrm{NMCy2}}(\mathbf{a})} \rho_\pi(\mathbf{z}) = \sum_{0 \le \mathbf{b} \le \mathbf{a}} (-1)^{\sum_{i \notin H} b_i} I_{n-\|\mathbf{b}\|}(\mathbf{z}) \prod_i \binom{a_i}{b_i} I_{b_i}(\sigma_i \mathbf{z})$$

holds.

34 Further directions

1. Find explicit combinatorial bijections.
2. Put more statistics:
   - Number of adjacent appearances of the same color in the NMCy model.
   - Number of fixed colors in NFiD or NFiA models.
3. q-versions?

References

[1] Guo-Niu Han and Guoce Xin. Permutations with extremal number of fixed points. *Journal of Combinatorial Theory, Series A*, 116:449–459, 2009.

[2] Niklas Eriksen, Ragnar Freij, Johan Wästlund. Enumeration of derangements with descents in prescribed positions. *The Electornic Journal of Combinatorics* 16 (2009), #R32.

[3] Dongsu Kim and Seunghyun Seo, Counting derangements with ascents and descents in given positions, in preparation.

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