

# Enumeration of Wilf Classes in $S_n \wr C_r$

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## Preliminary Definitions

$S_n^{(r)} = S_n \wr C_r$  (the wreath product)

- $S_n$  is the symmetric group.
- $C_r$  is the cyclic group of order  $r$  containing  $\{1, 2, \dots, r\}$

Colored Permutations,  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$

- Each symbol  $1, 2, \dots, n$  appears once.
- Each is colored by one of the colors  $\{1, 2, \dots, r\}$
- An example:  $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$

$$\phi \in S_3 \wr C_2$$

$$\phi \in S_3 \wr C_7$$

## Preliminary Definitions

Certain wreath products are isomorphic to other groups.

- $S_n^{(1)} \cong S_n$
- $S_n^{(2)} \cong B_n$

The absolute value notation is defined as follows:

- $|\phi| = (|\phi_1|, |\phi_2|, \dots, |\phi_n|)$   
where  $|\phi_j|$  is  $\phi_j$  with its color removed.
- For example, let  $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$ .  
Then  $|\phi| = (1, 3, 2)$

The cardinality of  $S_n^{(r)}$  is  $n!r^n$ .

## Pattern Matching Condition

(Used by Mansour)

Let  $\phi = (\tau_1^{(s_1)}, \dots, \tau_k^{(s_k)}) \in \mathcal{S}_k^{(r)}$

Let  $\psi = (\alpha_1^{(v_1)}, \dots, \alpha_n^{(v_n)}) \in \mathcal{S}_n^{(q)}$

$\psi$  contains  $\phi$  if  $\exists 1 \leq i_1 < \dots < i_k \leq n$  such that

- $(\alpha_{i_1}, \dots, \alpha_{i_k})$  is order-isomorphic to  $|\phi|$ .  
i.e.  $\text{red}((\alpha_{i_1}, \dots, \alpha_{i_k})) = |\phi|$
- $v_{i_j} = s_j$  for all  $j = 1, 2, \dots, k$ .

otherwise  $\psi$  avoids  $\phi$ .

$$Av_{n,r}(\phi) := \{\psi \in \mathcal{S}_n^{(r)} \mid \psi \text{ avoids } \phi\}$$

## An example of pattern containment

- Let  $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$
- Let  $\psi = (4^{(2)}, 2^{(1)}, 1^{(2)}, 5^{(2)}, 3^{(1)})$

$\psi$  contains  $\phi$

## An example of pattern containment

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$\psi$  contains  $\phi$

- $red((253)) = (132)$

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- Let  $\psi = (4^{(2)}, 2^{(1)}, 1^{(2)}, 5^{(2)}, 3^{(1)})$

$\psi$  contains  $\phi$

- $red((253)) = (132)$
- The colors match those of  $\phi$ .

## An example of pattern avoidance

- Let  $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$
- Let  $\sigma = (4^{(2)}, 2^{(1)}, 1^{(2)}, 5^{(2)}, 3^{(2)})$

$\sigma$  avoids  $\phi$

## An example of pattern avoidance

- Let  $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$
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$\sigma$  avoids  $\phi$

- $red((253)) = (132)$

## An example of pattern avoidance

- Let  $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$
- Let  $\sigma = (4^{(2)}, 2^{(1)}, 1^{(2)}, 5^{(2)}, 3^{(2)})$

$\sigma$  avoids  $\phi$

- $red((253)) = (132)$
- The colors do NOT match those of  $\phi$ .

## An example of pattern avoidance

- Let  $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$
- Let  $\sigma = (4^{(2)}, 2^{(1)}, 1^{(2)}, 5^{(2)}, 3^{(2)})$

$\sigma$  avoids  $\phi$

- $red((153)) = (132)$
- The colors do NOT match those of  $\phi$ .

## Collections of Colored Patterns

- Let  $T$  be an arbitrary finite collection of colored patterns.
- $\psi$  avoids  $T$  if  $\psi$  avoids all  $\phi \in T$
- Let  $T_1, T_2$  be two subsets of colored patterns.
- $T_1, T_2 \subset S_n^{(r)}$  are in the same Wilf class if  $|Av_{n,r}(T_1)| = |Av_{n,r}(T_2)|$  for all  $n > 0$  and  $r > 0$  (except possibly the first few terms).
- In this case  $T_1$  and  $T_2$  are said to be Wilf equivalent.

# Patterns of Length Two

## Theorem

(Mansour 2001)

- For all  $\phi \in S_2^{(r)}$ ,  $|Av_{n,r}(\phi)| = \sum_{j=0}^n j!(r-1)^j \binom{n}{j}^2$
- The number of Wilf classes for two patterns of length 2 using  $r$  colors is one for  $r = 1$ , four for  $r = 2$  and six for  $r \geq 3$ .

## Our Question

We study colored patterns of length three. We seek to enumerate the size of avoidance classes and Wilf classes of paired patterns.

Our Methods:

- Computer programs in Maple and Python.
- Bijections between patterns to prove Wilf equivalence.
- Bijections to known sequences in order to enumerate.
- Brute force enumeration (in the easiest cases).

## Our Programs

- Generate wreath product groups up to  $S_8 \wr C_2$
- Check the avoidance or containment of patterns using various pattern matching conditions.
- Enumerate avoidance or containment classes.
- Check for Wilf equivalence between patterns.
- Identify and enumerate Wilf classes.

## Bijections between patterns

Three maps were used by West and Mansour to prove the Wilf equivalence of colored patterns of length 2. We work with pattern pairs in  $S_3 \wr C_2$  and have developed another map which we call Inverse.

- Reversal
- Permute colors
- Complement permutation
- Inverse

## Definition

The following defines the Reversal map (R).

$$\phi = (\tau_1^{(s_1)}, \tau_2^{(s_2)}, \dots, \tau_k^{(s_k)})$$

$$R(\phi) = (\tau_k^{(s_k)}, \tau_{k-1}^{(s_{k-1})}, \dots, \tau_1^{(s_1)})$$

$$T_1 = \begin{pmatrix} 1^{(1)}, 2^{(2)}, 3^{(1)} \\ 2^{(1)}, 1^{(2)}, 3^{(2)} \end{pmatrix} \quad T_2 = \begin{pmatrix} 3^{(1)}, 2^{(2)}, 1^{(1)} \\ 3^{(2)}, 1^{(2)}, 2^{(1)} \end{pmatrix}$$

## Permute Colors

### Definition

The bijection entitled permute colors (PC) does not modify the underlying permutation of  $S_n$ . Instead, it deals only with the colors. These colors are permuted in some way as to identify a new pattern.

$$\phi = (\tau_1^{(s_1)}, \dots, \tau_k^{(s_k)}).$$

If  $\mu$  is any permutation, then  $PC(\phi) = \tau_1^{(\mu(s_1))}, \tau_2^{(\mu(s_2))}, \dots, \tau_k^{(\mu(s_k))}$ .

For example, let the permutation be defined by  $\mu : (12) \rightarrow (21)$

$$T_1 = \begin{pmatrix} 1^{(1)}, 2^{(2)}, 3^{(1)} \\ 2^{(1)}, 1^{(2)}, 3^{(2)} \end{pmatrix} T_3 = \begin{pmatrix} 1^{(2)}, 2^{(1)}, 3^{(2)} \\ 2^{(2)}, 1^{(1)}, 3^{(1)} \end{pmatrix}$$

# Complement Permutation

## Definition

Complement permutation (CP) does not modify the colors nor their order. It deals only with  $|\phi|$ .

The bijection maps  $\tau_i$  to  $n + 1 - \tau_i$ .

$$\phi = (\tau_1^{(s_1)}, \dots, \tau_k^{(s_k)})$$

$$CP(\phi) = ((k + 1 - \tau_1)^{(s_1)}, \dots, (k + 1 - \tau_k)^{(s_k)})$$

$$T_1 = \left( \begin{array}{c} 1^{(1)}, 2^{(2)}, 3^{(1)} \\ 2^{(1)}, 1^{(2)}, 3^{(2)} \end{array} \right) T_4 = \left( \begin{array}{c} 3^{(1)}, 2^{(2)}, 1^{(1)} \\ 2^{(1)}, 3^{(2)}, 1^{(2)} \end{array} \right)$$

## Inverse

### Definition

The bijection entitled inverse ( $I$ ) is similar to the inverse of a permutation in  $S_n$ . This bijection maps  $\tau_i^{(s_i)}$  to  $\tau^{-1}(i)^{(s_{\tau^{-1}(i)})}$

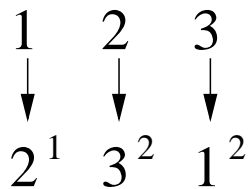
$$\phi = (\tau_1^{(s_1)}, \dots, \tau_k^{(s_k)})$$

$$I(\phi) = (\tau^{-1}(1)^{(s_{\tau^{-1}(1)})}, \dots, \tau^{-1}(k)^{(s_{\tau^{-1}(k)})})$$

$$T_1 = \left( \begin{array}{c} 1^{(1)}, 2^{(2)}, 3^{(1)} \\ 2^{(1)}, 1^{(2)}, 3^{(2)} \end{array} \right) T_5 = \left( \begin{array}{c} 1^{(1)}, 2^{(2)}, 3^{(1)} \\ 2^{(2)}, 1^{(1)}, 3^{(2)} \end{array} \right)$$

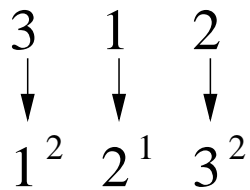
# Inverse

Figure



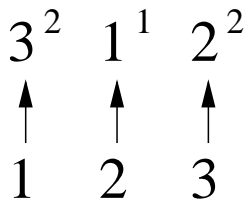
# Inverse

Figure



# Inverse

Figure



# Proofs

To prove that these maps preserve Wilf equivalence, the following definition and lemma are required.

## Definition

We say a bijection,  $g$ , is *well defined on subsequences* if and only if for every sequence  $\psi$  and every subsequence  $\phi$  of  $\psi$ ,  $g(\phi)$  is well defined and  $g(\phi)$  is a subsequence of  $g(\psi)$ .

## An example and a note

For example,

Suppose  $g$  is the bijection which flips 1 and 2 in a permutation.

- If  $\psi = 1234$  and  $\phi = 12$  then  $g(\psi) = 2134$  and  $g(\phi) = 21$ .
- If  $\tau = 13$  and we are required to determine  $g(\tau)$ , this is not well defined on subsequences since  $\tau$  contains a 1 but no 2.

**Note** The previous example could be modified so that  $g(\tau) = 13$ , or that  $g$  fixes subsequences that don't contain both a 1 and a 2. However, we can adjust  $\psi$  to be  $\psi = 1342$  so that  $g(\tau)$  is not contained in  $g(\psi)$ .

## A lemma and a theorem

### Lemma

*If  $g$  is well defined on subsequences then  $g$  preserves Wilf equivalence.*

### Theorem

*The inverse map preserves Wilf equivalence.*

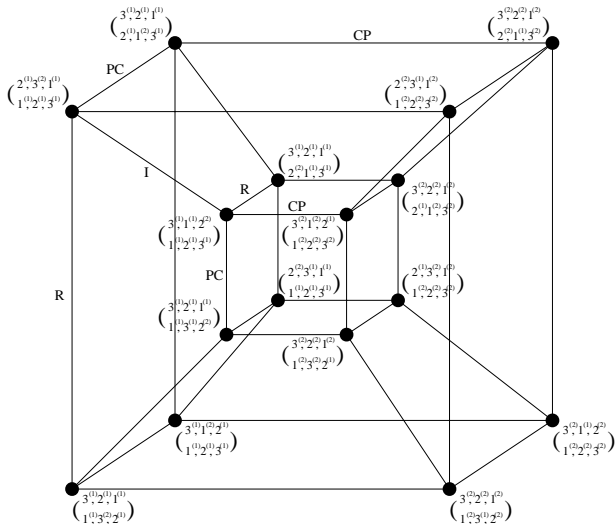
## Forming Blocks

Using these four maps we can lump certain pattern pairs together and prove their Wilf equivalence. This forms "blocks" within the Wilf classes identified by the programs.

Many of the projected Wilf classes contain several disjoint blocks. The following pages contain examples of such classes.

# A complete Wilf class

Figure

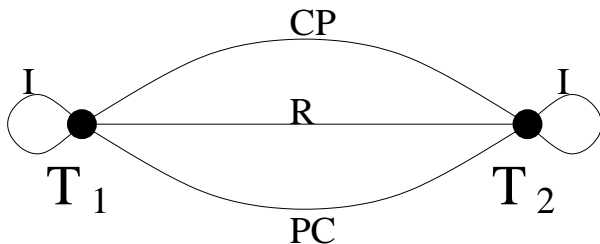


## A complete Wilf class

Figure

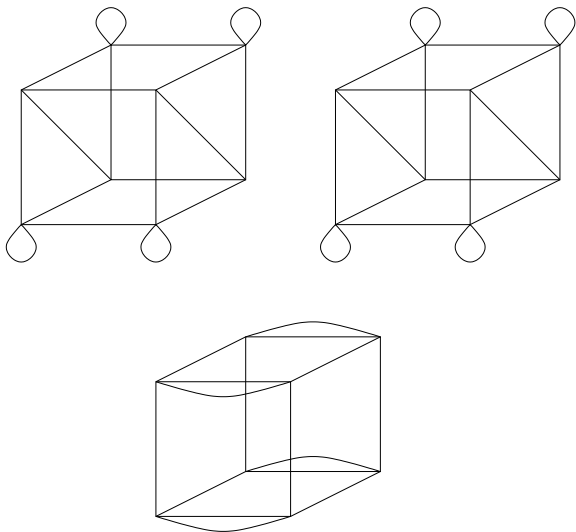
$$T_1 = \begin{pmatrix} 3^1 & 2^2 & 1^1 \\ 3^2 & 2^1 & 1^2 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 1^1 & 2^2 & 3^1 \\ 1^2 & 2^1 & 3^2 \end{pmatrix}$$



## A fragmented Wilf class

Figure



## A conjecture on Wilf equivalence

The following are representatives from 7 blocks that form one projected Wilf class.

We conjectured that these pattern pairs are Wilf equivalent.

$$\begin{aligned} & \left( \begin{array}{c} 1^{(1)}, 2^{(1)}, 3^{(1)} \\ 1^{(2)}, 2^{(2)}, 3^{(2)} \end{array} \right) \left( \begin{array}{c} 1^{(1)}, 2^{(1)}, 3^{(1)} \\ 1^{(2)}, 3^{(2)}, 2^{(2)} \end{array} \right) \left( \begin{array}{c} 1^{(1)}, 2^{(1)}, 3^{(1)} \\ 2^{(2)}, 3^{(2)}, 1^{(2)} \end{array} \right) \\ & \left( \begin{array}{c} 1^{(1)}, 2^{(1)}, 3^{(1)} \\ 3^{(2)}, 2^{(2)}, 1^{(2)} \end{array} \right) \left( \begin{array}{c} 1^{(1)}, 3^{(1)}, 2^{(1)} \\ 2^{(2)}, 1^{(2)}, 3^{(2)} \end{array} \right) \left( \begin{array}{c} 1^{(1)}, 3^{(1)}, 2^{(1)} \\ 3^{(2)}, 1^{(2)}, 2^{(2)} \end{array} \right) \\ & \left( \begin{array}{c} 1^{(1)}, 3^{(1)}, 2^{(1)} \\ 1^{(2)}, 3^{(2)}, 2^{(2)} \end{array} \right) \end{aligned}$$

# Brute force Enumeration of Unicolored Patterns

## Definition

A *unicolored* pattern is a pattern  $\phi$  in  $S_n^{(r)}$  for which every element of  $|\phi|$  is assigned the same color.

# Brute force Enumeration of Unicolored Patterns

## Theorem

Let  $\phi_1, \phi_2$  be unicolored patterns in  $S_3 \wr C_2$  and let them be colored with 1s and 2s respectively. Note that  $S_i(|\phi_1|) = C_i$  and  $S_j(|\phi_2|) = C_j$  where  $C_i$  and  $C_j$  denote the  $i$ th and  $j$ th Catalan numbers.

$$|\text{Av}_{n,r}(\phi_1, \phi_2)| = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i}^2 \binom{n-i}{j}^2 (n-i-j)! (r-2)^{n-i-j} C_i C_j$$



$$|\text{Av}_{n,r}(\phi_1, \phi_2)| = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i}^2 \binom{n-i}{j}^2 (n-i-j)! (r-2)^{n-i-j} C_i C_j$$

- Chooses the positions of the elements colored 1 in the colored permutation and chooses which elements they shall be.

Figure

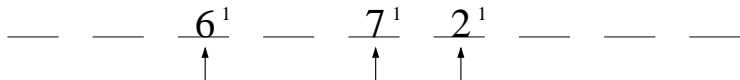
2 6 7



$$|\text{Av}_{n,r}(\phi_1, \phi_2)| = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i}^2 \binom{n-i}{j}^2 (n-i-j)! (r-2)^{n-i-j} C_i C_j$$

- Orders these numbers in these positions in such a way as to ensure that they avoid  $|\phi_1|$ .

Figure

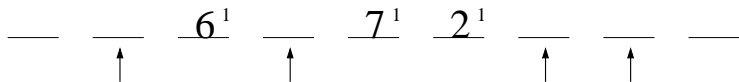


$$|\text{Av}_{n,r}(\phi_1, \phi_2)| = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i}^2 \binom{n-i}{j}^2 (n-i-j)! (r-2)^{n-i-j} C_i C_j$$

- Chooses the positions of the elements colored 2 in the colored permutation and chooses which elements they shall be.

Figure

1 3 5 8





$$|\text{Av}_{n,r}(\phi_1, \phi_2)| = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i}^2 \binom{n-i}{j}^2 (n-i-j)! (r-2)^{n-i-j} C_i C_j$$

- Arranges those elements of the permutation colored some number  $p \geq 3$ .

Figure

4   1<sup>2</sup>   6<sup>1</sup>   8<sup>2</sup>   7<sup>1</sup>   2<sup>1</sup>   3<sup>2</sup>   5<sup>2</sup>   9

$$|\text{Av}_{n,r}(\phi_1, \phi_2)| = \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i}^2 \binom{n-i}{j}^2 (n-i-j)! (r-2)^{n-i-j} C_i C_j$$

- Chooses colors for those remaining elements.

Figure

4<sup>3</sup>   1<sup>2</sup>   6<sup>1</sup>   8<sup>2</sup>   7<sup>1</sup>   2<sup>1</sup>   3<sup>2</sup>   5<sup>2</sup>   9<sup>5</sup>

This formula can also be written in terms of a hypergeometric series.

$$|Av_{n,r}(T)| = \sum_{i=0}^n \binom{n}{i}^2 (n-i)! (r-2)^{n-i} C_i {}_2F_2 \left( \begin{matrix} \frac{1}{2}, -n+i \\ 1, 2 \end{matrix} ; \frac{-4}{r-2} \right)$$

## Theorem

Let  $T$  denote a set of patterns in  $S_p \wr C_k$  where each element  $\phi$  is uni-colored. Let  $T_j$  denote the subset of  $T$  which are uni-colored  $j$ . Then for a general element in  $S_n \wr C_r$ ,  $n \geq p$ ,  $r \geq k$ , let  $i_h$ ,  $h \in \mathbb{N}$ , denote the number of permutation elements colored  $h$  for  $1 \leq h \leq k$ . Let  $i_{k+1}$  denote the number of elements in the permutation colored some number not included in the patterns in  $T$ .  $S_p(|T_j|)$  is the number of permutations in the symmetric group of order  $p$  that avoid the pattern  $|T_j|$ .

$$|Av_{n,r}(T)| = \sum_{\binom{i_1, \dots, i_{k+1}}{\sum_{j=1}^{k+1} i_j = n}} (r-k)^{i_{k+1}} \binom{n}{i_1, \dots, i_k, i_{k+1}}^2 (i_{k+1})! \prod_{j=1}^k S_{i_j}(|T_j|)$$

## What we are working on now

Patterns with one element colored 2 and the rest colored 1.

- Recursions (inserting  $n$  at active sites)
- Generating trees (forests). Problem: pairs and triples of labels vary from 1 to  $r$ .

# Recursions

## Definition

Let  ${}^p A_{n,r}(\phi)$  represent the set of elements in  $A_{n,r}(\phi)$  where the number  $n$  is colored by  $p$ .

## Theorem

$$|A_{n,r}(\phi)| = \sum_{p=1}^r |{}^p A_{n,r}(\phi)|$$

## An example recursion

For the recursion, let  $T = \begin{pmatrix} 1^{(1)}, 2^{(1)}, 3^{(1)} \\ 1^{(1)}, 2^{(1)}, 3^{(2)} \end{pmatrix}$

### Definition

Let  $Av_{n,r}^{i,j}(T)$  represent the set of elements in  $Av_{n,r}(T)$  where  $i$  identifies the position of the first *element* colored 1 and  $j$  identifies the position of the first *increase* colored 1.  $j \geq i$ . If there are no elements colored 1 then  $i = n + 1$  and  $j = n + 1$ . If there is a first element colored 1 but no increase colored 1, then  $j = n + 1$ . If  $i = j \neq n + 1$  then  $Av_{n,r}^{i,j}(T) = 0$ .

## Initial Conditions

- $|{}^1Av_{2,r}^{1,2}(T)| = 1$
  - $|{}^1Av_{2,r}^{1,3}(T)| = r$
  - $|{}^1Av_{2,r}^{2,3}(T)| = r - 1$
  - $|{}^1Av_{2,r}^{3,3}(T)| = 0$
- for  $p \geq 2$
- $|{}^pAv_{2,r}^{1,2}(T)| = 0$
  - $|{}^pAv_{2,r}^{1,3}(T)| = 1$
  - $|{}^pAv_{2,r}^{2,3}(T)| = 1$
  - $|{}^pAv_{2,r}^{3,3}(T)| = 2(r - 1)$

## Recursion

Inserting  $n^1$  into a permutation in  $S_{n-1}^r$  that already avoids  $T$ .

- $|{}^1A_{n,r}^{i,j}(T)| = \sum_{a=i}^{j-1} |A_{n-1,r}^{a,j-1}(T)| + \sum_{b=j}^n |A_{n-1,r}^{i,b}(T)|$

Inserting  $n^2$ .

- $|{}^2A_{n,r}^{i,j}(T)| = (i-1)|A_{n-1,r}^{i-1,j-1}(T)| + (j-i-1)|A_{n-1,r}^{i,j-1}(T)|$

Inserting  $n^p$   $p > 2$ .

- $|{}^pA_{n,r}^{i,j}(T)| = (i-1)|A_{n-1,r}^{i-1,j-1}(T)| + (j-i-1)|A_{n-1,r}^{i,j}(T)|$

## Recursion

### Theorem

$$\begin{aligned} |Av_{n,r}(T)| &= \sum_{p=1}^r \sum_{i=1}^{n+1} \sum_{j=i}^{n+1} \\ &\quad \left( \sum_{a=i}^{j-1} |{}^p Av_{n-1,r}^{a,j-1}(T)| + \sum_{b=j}^n |{}^p Av_{n-1,r}^{i,b}(T)| \right. \\ &\quad + (i-1)(r-1) |{}^p Av_{n-1,r}^{i-1,j-1}(T)| \\ &\quad + (j-i-1)(r-1) |{}^p Av_{n-1,r}^{i,j-1}(T)| \\ &\quad \left. + (r-2)(n-j) |{}^p Av_{n-1,r}^{i,j}(T)| \right) \end{aligned}$$

## Another conjecture on Wilf equivalence

The following are representatives from 3 blocks that form a projected Wilf class. These blocks form the image shown before. We have not proved this conjecture.

### Conjecture

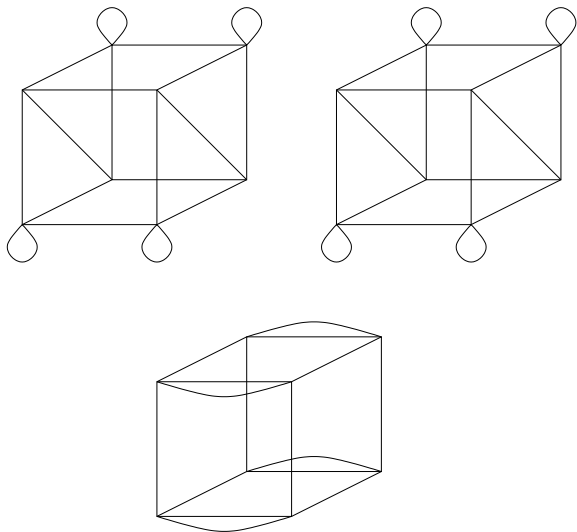
*We conjecture that the pattern pairs*

$$\left( \begin{array}{c} 3^{(1)}, 2^{(1)}, 1^{(2)} \\ 3^{(1)}, 2^{(2)}, 1^{(1)} \end{array} \right), \left( \begin{array}{c} 2^{(1)}, 3^{(2)}, 1^{(1)} \\ 2^{(2)}, 3^{(1)}, 1^{(1)} \end{array} \right), \text{ and } \left( \begin{array}{c} 2^{(1)}, 3^{(2)}, 1^{(1)} \\ 2^{(2)}, 1^{(1)}, 3^{(2)} \end{array} \right)$$

*are Wilf equivalent.*

# The fragmented unproven Wilf class

Figure



## Summary

- Paired colored patterns of length three.
- Maps to form blocks.
- Enumeration of uni-colored patterns.
- Recursions for patterns with one 2.

Thank you!