Permutations Invariant Under *rc* and Avoiding 4321

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Symmetries of Permutations

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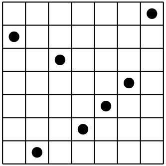

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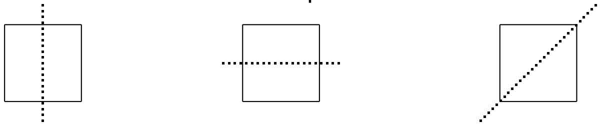

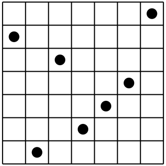

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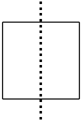

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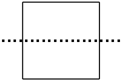

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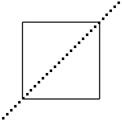

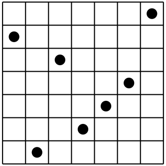

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π contains/avoids σ if and only if π^f contains/avoids σ^f

Subgroups of D_4

- $H_0 = \{e\}$
- $H_1 = \{e, rc\}$
- $H_2 = \{e, i, rc, rci\}$
- $H_3 = \{e, rc, ri, ci\}$
- $H_{4a} = \{e, i\}$
- $H_{4b} = \{e, rci\}$
- $H_5 = \{e, r\}$
- $H_6 = \{e, c\}$
- $H_7 = \{e, r, c, rc\}$
- $H_8 = D_4$

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General Question

How many permutations in S_n

- *avoid a given set R of patterns and*
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Three other groups give new, nontrivial answers.

Some Known Enumerations

Theorem (Guibert and Pergola, 2000)

$$|I_{2n}^{rc}(2143)| = \sum_{i=0}^{n} \frac{n!}{(n-i)!\lfloor i/2 \rfloor! \lceil i/2 \rceil!}.$$

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Question

What is $|S_{2n}^{rc}(4321)|$*?*

The Generating Tree for S_{2n}^{rc}

$$\pi \in S_{2n}^{rc}$$

To get the parent of π

- remove 1 and $2n$
- decrease every entry by 1.

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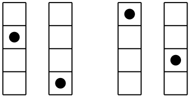

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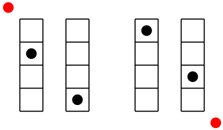

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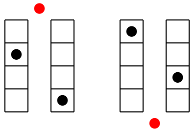

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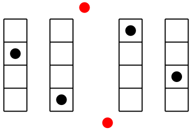

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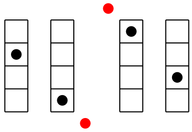

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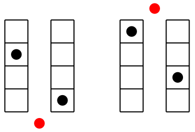

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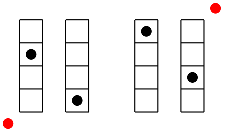

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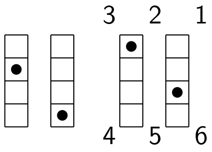

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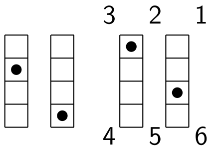

Each $\pi \in S_{2n}^{rc}$ has

- $2n + 2$ sites
- $2n + 2$ children in S_{2n+2}^{rc}.

The Generating Tree for $S_{2n}^{rc}(4321)$

The label for $\pi \in S_{2n}^{rc}$ is (i, j, k) where

- i is the number of active sites for $2n + 2$ (top row)
- j is the number of sites where $2n + 2$ does not produce a 321
- k is the number of active sites for 1 (bottom row)

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Two of the Five Transition Rules:

$$\begin{aligned}(i, j, 0) \rightarrow (i + 1, j + 1, 0),\\ &(i + 1, 2, 0), (i + 1, 3, 0), \ldots, (i + 1, j, 0),\\ &(j + 1, j, 0), (j + 2, j, 0), \ldots, (i, j, 0)\end{aligned}$$

$$\begin{aligned}(i, i, 1) \rightarrow (i + 1, i + 1, 1),\\ &(i + 1, 2, 1), (i + 1, 3, 1), \ldots, (i + 1, i - 1, 1),\\ &(i + 1, i, 2), (i + 1, i + 1, 1)\end{aligned}$$

One of the Functional Equations

$$Q(x,u,v,w) = \sum_{\substack{i>j \\ 0\le k<i-j+1}} x^{\frac{|\pi|}{2}} u^i v^j w^k$$

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$$\left(1 - \frac{xuv^2}{v-1} - \frac{xuw}{w-1}\right) Q(x,u,v,w) =$$

$$\frac{xu}{u-1} Q(x,u,v,1) - \frac{xuv^2}{v-1} Q(x,u,1,w)$$

$$-\frac{xu}{u-1} Q(x,1,uv,1) - \frac{xuw}{w-1} Q(x,u,v,1) + R(x,u,v,w)$$

The RSK Correspondence

$$\pi \mapsto (P(\pi), Q(\pi))$$

Example

$$\pi = 52413$$

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5	1

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2	1
5	2

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2	4
5	

1	3
2	

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1	4
2	
5	

1	3
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4	

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$P(\pi)$

1	3
2	5
4	

$Q(\pi)$

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1	3
2	4
5	

$P(\pi)$

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4	

$Q(\pi)$

Theorem (Schensted)

The length of the longest decreasing subsequence in π is the number of rows in $P(\pi)$.

Evacuation

$$ev : standard\ \ tableaux \longrightarrow standard\ \ tableaux$$

$$ev \circ ev = identity$$

Example

1	3
2	5
4	

P

$ev(P)$

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4	

P

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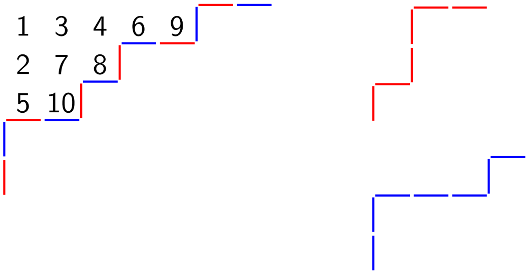
3	4
5	

$ev(P)$

Theorem

$\pi^{rc} = \pi$ *if and only if* $ev(P(\pi)) = P(\pi)$ *and* $ev(Q(\pi)) = Q(\pi)$.

Factoring Self-Evacuating Tableaux



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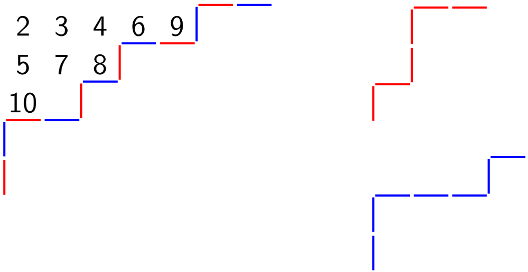

At each step we

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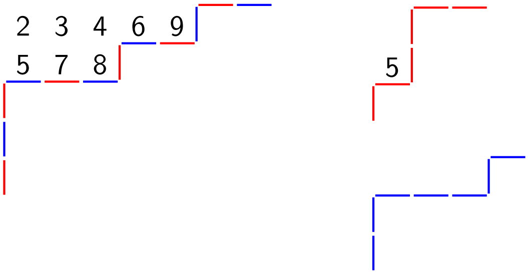

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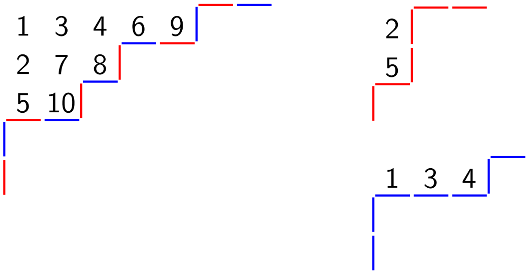

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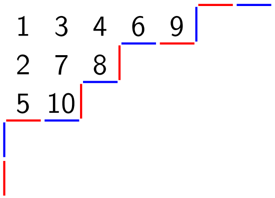

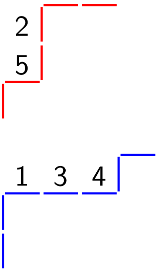

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Fact

If P has $2n$ entries and at most 3 rows then

- *the entries of P_r and P_b partition $[n]$;*
- *P_r has at most 2 rows;*
- *P_b has at most 1 row.*

Putting it all together

$$\pi \mapsto (P(\pi), Q(\pi)) \mapsto (P_r(\pi), P_b(\pi), Q_r(\pi), Q_b(\pi)) \mapsto E_1, E_2, \pi_r, \pi_b$$

- $\pi \in S_{2n}^{rc}(4321)$

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- $\pi \in S_{2n}^{rc}(4321)$
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- $\pi \in S_{2n}^{rc}(4321)$
- $P(\pi)$, $Q(\pi)$ have $2n$ entries and at most 3 rows.
- P_r, P_b have entries $[n]$ and Q_r, Q_b have entries $[n]$.
- P_r, Q_r each has at most 2 rows and P_b, Q_b each has at most 1 row.

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- $E_1, E_2 \subseteq [n]$ and $|E_1| = |E_2|$.
- $\pi_r \in S_{|E_1|}(321)$ and $\pi_b \in S_{|E_1|}(21)$.

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Theorem (E,2010)

$$|S_{2n+1}^{rc}(4321)| = |S_{2n}^{rc}(4321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j,$$

Results for $S_n^{rc}(k \ldots 321)$

Theorem (E,2010)

For all $k \geq 2$ and all $n \geq 0$ we have

$$|S_{2n}^{rc}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j}^2 \left| S_j \left(\left\lceil \frac{k+1}{2} \right\rceil \ldots 21 \right) \right| \left| S_{n-j} \left(\left\lfloor \frac{k+1}{2} \right\rfloor \ldots 21 \right) \right|.$$

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Corollary

$$|S_{2n}^{rc}(54321)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j C_{n-j},$$

Results for Involutions

Theorem (E, 2010)

For all $k \geq 2$ *and all* $n \geq 0$ *we have*

$$|I_{2n}^{rc}(k \ldots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j \left(\left\lceil \frac{k+1}{2} \right\rceil \ldots 21 \right) \right| \left| I_{n-j} \left(\left\lfloor \frac{k+1}{2} \right\rfloor \ldots 21 \right) \right|.$$

Results for Involutions

Theorem (E, 2010)

For all $k \geq 2$ and all $n \geq 0$ we have

$$|I_{2n}^{rc}(k\dots 21)| = \sum_{j=0}^{n} \binom{n}{j} \left| I_j \left(\left\lceil \frac{k+1}{2} \right\rceil \dots 21 \right) \right| \left| I_{n-j} \left(\left\lfloor \frac{k+1}{2} \right\rfloor \dots 21 \right) \right|.$$

Corollary

$$|I_{2n}^{rc}(54321)| = \binom{n}{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{\lfloor \frac{n+1}{2} \rfloor};$$

$$|I_{2n+1}^{rc}(54321)| = C_{n+1};$$

$$|I_{2n}^{rc}(7654321)| = \sum_{j=0}^{n} \binom{n}{j} M_j M_{n-j}.$$

Thank You!