Pattern and position based permutation statistics

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Motivi di permutazioni
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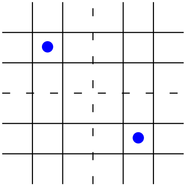
Patterns of length two

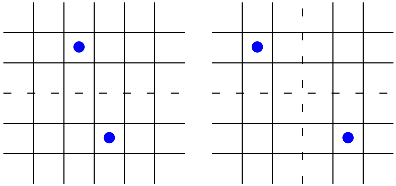
There is only one distribution for classical patterns of length 2, the **inversion** 21. Statistics with the same distribution are called **Mahonian**.

With **generalised patterns** (Babson-Steingrímsson), we have two distributions: inversions 2–1 and **descents** 21. Statistics with the same distribution as descents are called **Eulerian**.

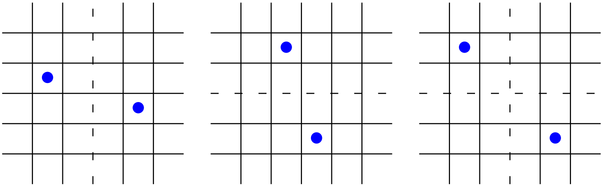
Doubly generalised patterns



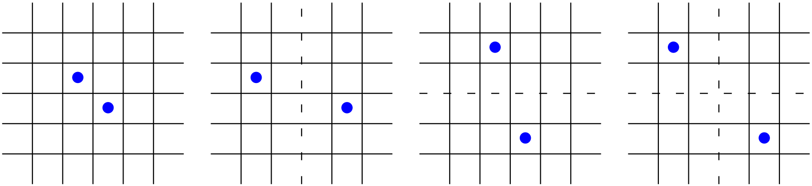
Doubly generalised patterns



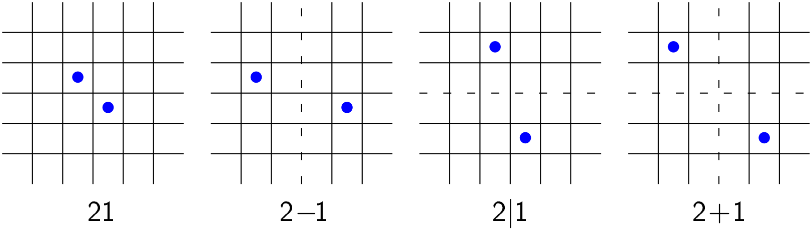
Doubly generalised patterns



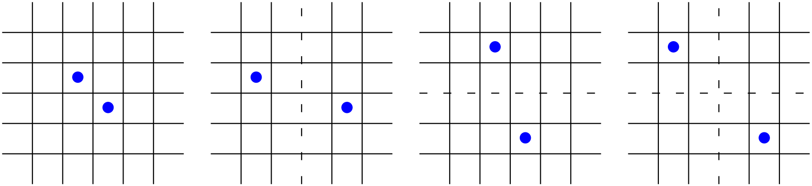
Doubly generalised patterns



Doubly generalised patterns



Doubly generalised patterns



21   2–1   2|1   2+1

The pattern 21 is called an **adjacency**.

Pattern and position statistics

Permutation statistics for $\mathfrak{S}_n$.

Pattern and position statistics

Permutation statistics for ${\mathfrak{S}_n}^0 = \{[\pi\ 0] : \pi \in \mathfrak{S}_n\}$.

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		Eulerian	Mahonian
Pattern	21 Adj	2|1 Des	2+1 Inv
Position	$=$ Fix	$\geq$ Exc	? ?

Pattern and position statistics

Permutation statistics for ${\mathfrak{S}_n}^0 = \{[\pi\ 0] : \pi \in \mathfrak{S}_n\}$.

		Eulerian	Mahonian
Pattern	21 Adj	2|1 Des	2+1 Inv
Position	$=$ Fix	$\geq$ Exc	? ?

Theorem

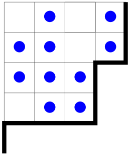
The bistatistics $\{\mathrm{Adj}, \mathrm{Des}\}$ *and* $\{\mathrm{Fix}, \mathrm{Exc}\}$ *on* $\mathfrak{S}_n^0$ *are equidistributed.*

Permutation tableaux

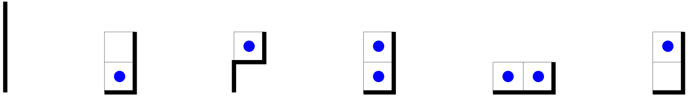
A **permutation tableau** is a filling of some cells in a Ferrers board (allowing zero parts) such that

(1-hinge) A cell must be filled if there is a filled cell to its left in the same row and a filled cell above it in the same column.

(column) Every column contains at least one filled cell.

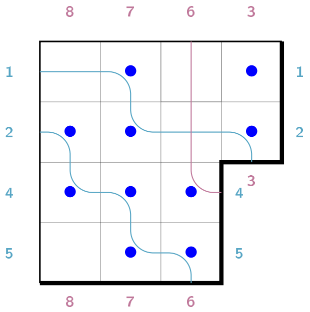


Permutations tableaux with n steps on the South-East border (bold) are in bijection with permutations on n elements.



Φ: From tableau to permutation

- Label South-East border with numbers 1 to n and distribute to rows and columns (entry labels).
- Start paths at North-West border, bouncing down and right at filled cells.
- If a path starts at label i and ends at label j, then $\pi(i) = j$.



Rows correspond to excedances and empty rows to fixed points.

Pattern based tableaux

We will prove the theorem by giving a bijection from permutations to tableaux such that rows correspond to descents and empty rows to adjacencies. Corteel gave a bijection with the first, but not the latter, property.

But we start with a simpler bijection!

Neighbour crossing partitions

A **neighbour crossing partition** on n elements in k parts is a function $u : \{0, \ldots, n\} \to k$ such that

- u restricted to $\{1, \ldots, n\}$ is surjective;
- $u(0) = k$;
- for $1 \leq m \leq k - 1$, $\min u^{-1}(m) < \max u^{-1}(m + 1)$.

The set of these is denoted $\mathrm{NeighCP}(n, k)$.

Neighbour crossing partitions, cont

Lemma

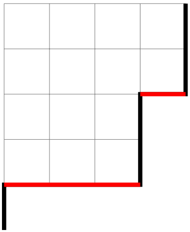
We have $|\mathrm{NeighCP}(n,k)| = A(n,k)$*, where* $A(n,k)$ *are the Eulerian numbers, that is the number of permutations* $\pi \in \mathfrak{S}_n$ *with* $k-1$ *descents.*

Consider permutation in $\mathfrak{S}_n^0$ with $k-1$ ascents. These are counted by $|\mathrm{NeighCP}(n,k)|$, since $u(m)$ tells which descending sequence the element m belongs to:

$$\pi = [7\ 5\ 2\ 1\ 6\ 3\ 4\ 0] \quad \Rightarrow \quad u = (3,1,1,2,3,1,2,1).$$

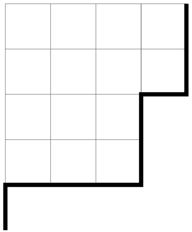
Bijection NeighCP$\to$ tableaux

$\pi = [7\ 9\ 3\ 8\ 2\ 4\ 6\ 5\ 1\ 0]$
$u = (5, 5, 3, 2, 4, 5, 5, 1, 3, 2)$



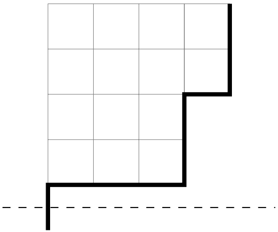
Bijection NeighCP$\to$ tableaux, cont

$u = (5, 5, 3, 2, 4, 5, 5, 1, 3, 2)$



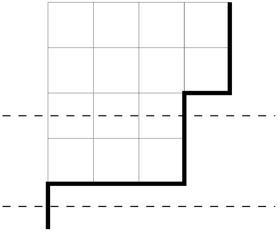
Bijection NeighCP→ tableaux, cont

$$u = (5, 5, 3, 2, 4, 5, 5, 1, 3, 2)$$
$$(5, 3, 2, 4, 5, 5, 1, 3, 2)$$



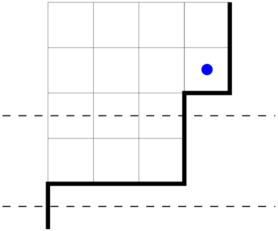
Bijection NeighCP$\rightarrow$ tableaux, cont

$$
\begin{aligned}
u = & (5, 5, 3, 2, 4, 5, 5, 1, 3, 2) \\
& (5, 3, 2, 4, 5, 5, 1, 3, 2) \\
& (5, 3, 2, 4, 5, 1, 3, 2)
\end{aligned}
$$



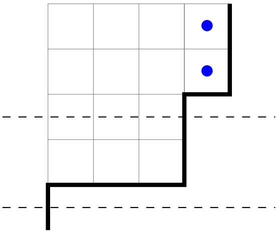
Bijection NeighCP$\rightarrow$ tableaux, cont

$$
\begin{aligned}
u = &(5, 5, 3, 2, 4, 5, 5, 1, 3, 2) \\
&(5, 3, 2, 4, 5, 5, 1, 3, 2) \\
&(5, 3, 2, 4, 5, 1, 3, 2) \\
&(5, 3, 2, 4, 5, 3, 1, 2)
\end{aligned}
$$



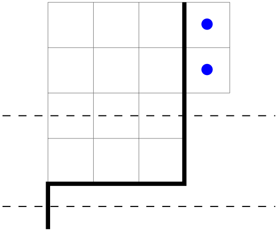
Bijection NeighCP$\rightarrow$ tableaux, cont

$$
\begin{aligned}
u = &(5, {\color{red}5}, 3, 2, 4, 5, 5, 1, 3, 2)\\
&(5, 3, 2, 4, 5, {\color{red}5}, 1, 3, 2)\\
&(5, 3, 2, 4, 5, {\color{blue}1}, 3, 2)\\
&(5, 3, 2, 4, 5, 3, {\color{blue}1}, 2)\\
&(5, 3, 2, 4, 5, 3, 2, {\color{green}1})
\end{aligned}
$$

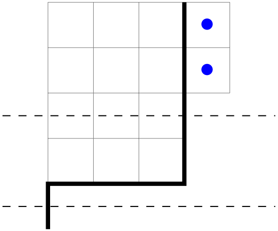


Bijection NeighCP$\to$ tableaux, cont

$$
\begin{aligned}
u = &(5, 5, 3, 2, 4, 5, 5, 1, 3, 2)\\
&(5, 3, 2, 4, 5, 5, 1, 3, 2)\\
&(5, 3, 2, 4, 5, 1, 3, 2)\\
&(5, 3, 2, 4, 5, 3, 1, 2)\\
&(5, 3, 2, 4, 5, 3, 2, 1)\\
&(4, 2, 1, 3, 4, 2, 1)
\end{aligned}
$$

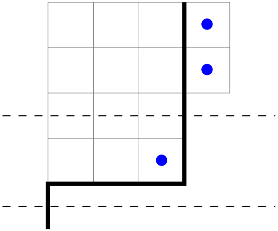


$u = (4, 2, 1, 3, 4, 2, 1)$



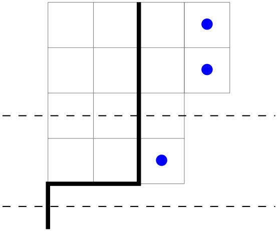
Bijection NeighCP$\to$ tableaux, cont

$$u = (4,2,1,3,4,2,1)$$
$$(4,2,1,4,3,2,1)$$



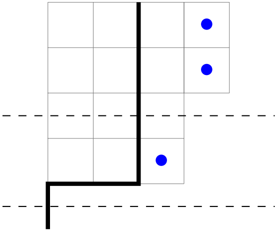
Bijection NeighCP $\rightarrow$ tableaux, cont

$$u = (4,2,1,3,4,2,1)$$
$$(4,2,1,4,3,2,1)$$
$$(3,2,1,3,2,1)$$



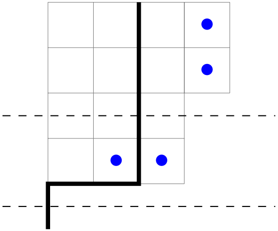
Bijection NeighCP→ tableaux, cont

$u = (3, 2, 1, 3, 2, 1)$



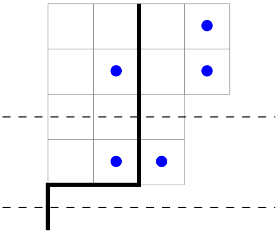
Bijection NeighCP→ tableaux, cont

$$u = (3, 2, 1, 3, 2, 1)$$
$$(3, 2, 3, 1, 2, 1)$$



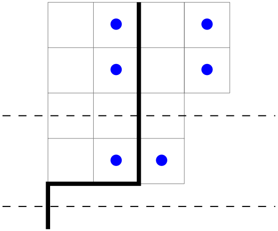
Bijection NeighCP$\longrightarrow$ tableaux, cont

$$
\begin{aligned}
u = &(3,2,1,3,2,1)\\
&(3,2,3,1,2,1)\\
&(3,2,3,2,1,2)
\end{aligned}
$$



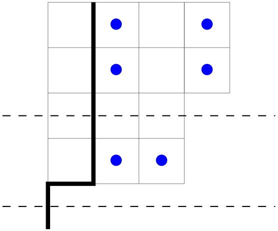
Bijection NeighCP$\to$ tableaux, cont

$$
\begin{aligned}
u = &(3, 2, 1, 3, 2, 1)\\
&(3, 2, 3, 1, 2, 1)\\
&(3, 2, 3, 2, 1, 2)\\
&(3, 2, 3, 2, 2, 1)
\end{aligned}
$$



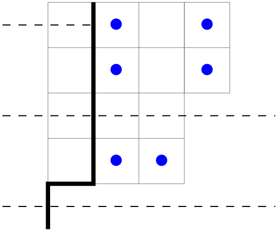
Bijection NeighCP→ tableaux, cont

$$
\begin{aligned}
u = &(3,2,1,3,2,1)\\
&(3,2,3,1,2,1)\\
&(3,2,3,2,1,2)\\
&(3,2,3,2,2,1)\\
&(2,1,2,1,1)
\end{aligned}
$$



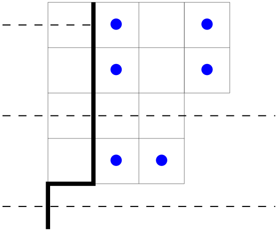
Bijection NeighCP→ tableaux, cont

$$
\begin{aligned}
u = &(3, 2, 1, 3, 2, 1)\\
&(3, 2, 3, 1, 2, 1)\\
&(3, 2, 3, 2, 1, 2)\\
&(3, 2, 3, 2, 2, 1)\\
&(2, 1, 2, 1, 1)\\
&(2, 1, 2, 1)
\end{aligned}
$$



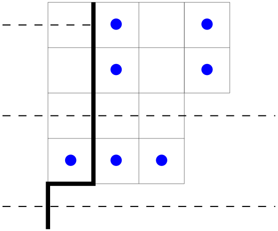
Bijection NeighCP$\longrightarrow$ tableaux, cont

$u = (2, 1, 2, 1)$



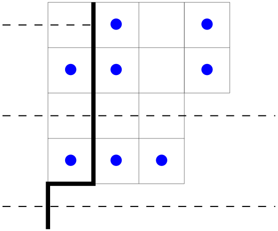
Bijection NeighCP$\to$ tableaux, cont

$$u = (2, 1, 2, 1)$$
$$(2, 2, 1, 2)$$



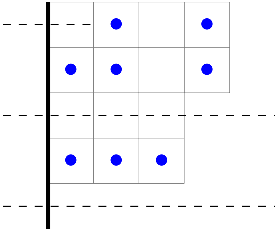
Bijection NeighCP → tableaux, cont

$$u = (2, {\color{blue}1}, 2, 1)$$
$$(2, 2, {\color{blue}1}, 2)$$
$$(2, 2, 2, {\color{green}1})$$



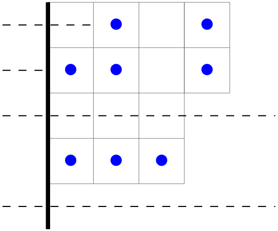
Bijection NeighCP$\rightarrow$ tableaux, cont

$$u = (2, 1, 2, 1)$$
$$(2, 2, 1, 2)$$
$$(2, 2, 2, 1)$$
$$(1, 1, 1)$$



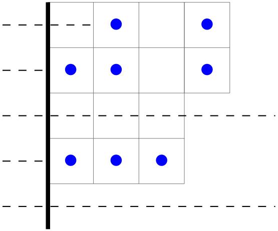
Bijection NeighCP$\to$ tableaux, cont

$$
\begin{aligned}
u = &(2, 1, 2, 1)\\
&(2, 2, 1, 2)\\
&(2, 2, 2, 1)\\
&(1, 1, 1)\\
&(1, 1)
\end{aligned}
$$



Bijection NeighCP→ tableaux, cont

$$
\begin{aligned}
u =& (2, 1, 2, 1) \\
& (2, 2, 1, 2) \\
& (2, 2, 2, 1) \\
& (1, 1, 1) \\
& (1, 1) \\
& (1)
\end{aligned}
$$



Inversions in position based tableaux

	20	19	18	17	16	14	12	9	8	4
1	●	●	NN	NN	NN	NN	NN	NN	NN	NN
2	EE	EE	EN	EN	EN	●	●	NN	NN	NN
3	●	●	●	NN	NN	●	●	●	NN	●
5	EE	EE	EN	EN	EN	EN	●	●	●	NE
6	EE	EE	EN	EN	●	●	●	●	●	NE
7	EE	EE	●	NN	●	●	●	●	●	NE
10	EE	EE	EE	EN	EN	●	●	NE	NE	NE
11	EE	EE	EE	EN	EN	●	●	NE	NE	NE
13	●	●	●	EN	●	●	NE	NE	NE	NE
15	EE	●	●	●	●	NE	NE	NE	NE	NE

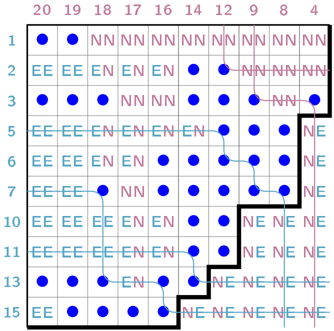
$$\#\mathrm{EE}(T) = A_{\mathrm{EE}}(\pi) = |\{(i,j) \mid i < j \leq \pi(j) < \pi(i)\}|$$

$$\#\mathrm{NN}(T) = A_{\mathrm{NN}}(\pi) = |\{(i,j) \mid \pi(j) < \pi(i) < i < j\}|$$

$$\#\mathrm{EN}(T) = A_{\mathrm{EN}}(\pi) = |\{(i,j) \mid i \leq \pi(i) < \pi(j) < j\}|$$

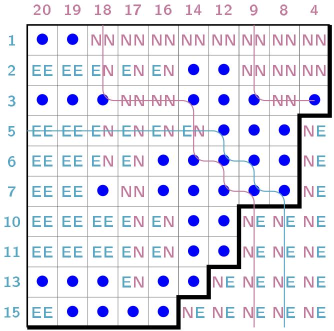
$$\#\mathrm{NE}(T) = A_{\mathrm{NE}}(\pi) = |\{(i,j) \mid \pi(i) < i < j \leq \pi(j)\}|$$

Inversions in position based tableaux



Two rows contribute an inversion for every EE.
Two columns contribute an inversion for every NN.

Inversions in position based tableaux



If a row index i is smaller than a column index j, then these positions contribute an inversion if the paths do not cross.

Inversions in position based tableaux

	20	19	18	17	16	14	12	9	8	4
1	●	●	NN	NN	NN	NN	NN	NN	NN	NN
2	EE	EE	EN	EN	EN	●	●	NN	NN	NN
3	●	●	●	NN	NN	●	●	●	NN	●
5	EE	EE	EN	EN	EN	EN	●	●	●	NE
6	EE	EE	EN	EN	●	●	●	●	●	NE
7	EE	EE	●	NN	●	●	●	●	●	NE
10	EE	EE	EE	EN	EN	●	●	NE	NE	NE
11	EE	EE	EE	EN	EN	●	●	NE	NE	NE
13	●	●	●	EN	●	●	NE	NE	NE	NE
15	EE	●	●	●	●	NE	NE	NE	NE	NE

If a columns index i is smaller than a row index j, we do not get an inversion. These are counted by NE.

Inversions in position based tableaux

	20	19	18	17	16	14	12	9	8	4
1	●	●	NN	NN	NN	NN	NN	NN	NN	NN
2	EE	EE	EN	EN	EN	●	●	NN	NN	NN
3	●	●	●	NN	NN	●	●	●	NN	●
5	EE	EE	EN	EN	EN	EN	●	●	●	NE
6	EE	EE	EN	EN	●	●	●	●	●	NE
7	EE	EE	●	NN	●	●	●	●	●	NE
10	EE	EE	EE	EN	EN	●	●	NE	NE	NE
11	EE	EE	EE	EN	EN	●	●	NE	NE	NE
13	●	●	●	EN	●	●	NE	NE	NE	NE
15	EE	●	●	●	●	NE	NE	NE	NE	NE

$$\mathrm{inv}(\pi) = \mathrm{exc}(\pi)(n - \mathrm{exc}(\pi)) - \mathsf{EN}(\mathcal{T}) - \mathsf{NE}(\mathcal{T}) + \mathsf{EE}(\mathcal{T}) + \mathsf{NN}(\mathcal{T}).$$

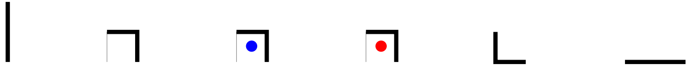
Splitting Babson-Steingrímsson descent-Mahonian equivalence classes

Let $\mathrm{Stat} = (13-2) + (21-3) + (32-1)$ (Babson-Steingrímsson).

Conjecture

The triple statistics $(\mathrm{Adj}, \mathrm{Des}, \mathrm{Stat})$ *is equidistributed with* $(\mathrm{Adj}, \mathrm{Des}, \mathrm{Maj})$.

Viennot's alternative tableaux are in bijection with $\mathfrak{S}_n$.



Conjecture

The statistic Astat *given by*

$$\mathrm{Astat}(\mathcal{T}) = \binom{\mathrm{rows}(\mathcal{T})+1}{2} + \text{red or blue cells} + \text{cells covered by red or blue cells}$$

is Mahonian. Further, $(\mathrm{rows}, \mathsf{Astat})$ *is equidistributed with* $(\mathsf{Des}, \mathsf{Maj})$ *and* $(\mathrm{emptyrows}, \mathrm{rows})$ *is equidistributed with* $(\mathsf{Adj}, \mathsf{Des})$ *on* $\mathfrak{S}_n$.

Questions?

- Domanda?
- Les questions?
- Frågor?