About non minimal permutations

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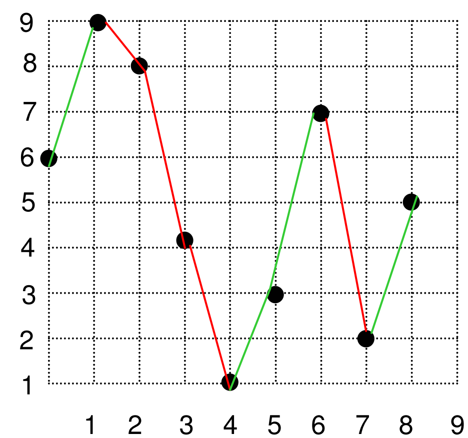
Outline of the talk

1. Definitions
2. Enumeration
3. Some statistics

Descents in permutations

Let $\sigma = \sigma(1)\ldots\ldots\ \sigma(n)$ in S_n, σ has a descent (resp. ascent) at position i, $1 \leq i \leq n-1$, iff $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) < \sigma(i+1)$)

$\sigma = 6\ 9\ 8\ 4\ 1\ 3\ 7\ 2\ 5$



Patterns in permutations

Let π in S_k, π is a pattern of $\sigma = \sigma(1) \ldots\ldots. \sigma(n)$ in S_n when it exists a sequence $i_{1.} \ldots. i_k$, $1 \leq i_{1.} < \ldots < i_k \leq n$, such that $\sigma(i_1) \ldots. \sigma(i_k)$ is order-isomorphic to π

1 2 3 4 is a pattern of *3 1 2 8 5 4 7 9 6*

The class of permutations avoiding the patterns $\pi_1 \ldots\ldots. \pi_j$ is denoted by $S(\pi_1 \ldots\ldots. \pi_j)$

1 4 2 5 6 3 is a permutation in *S(321)*

Minimal permutations with *d* descents: a first characterization

Minimal pemutation with $d = 2^p$ descents arise from biological motivations

Chaudhuri, Chen, Mihaescu, Rao, *On the tandem Duplication-random loss model of genome rearrangement*, SODA06

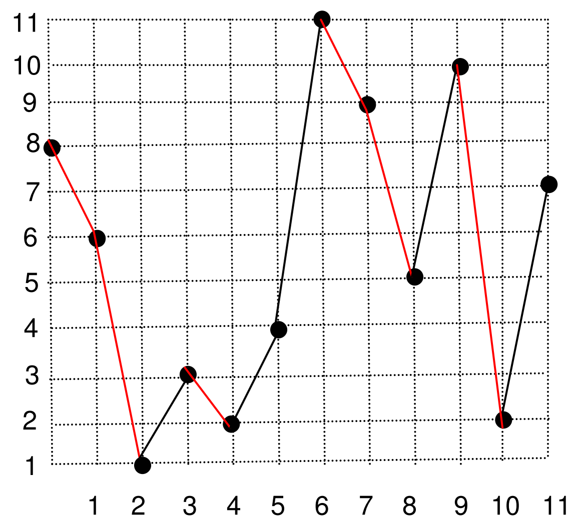
Bouvel, Rossin, *A variant of the tandem duplication-random loss model of genome rearrangement, TCS, 410 (8-10), pp. 847-858* (2009)

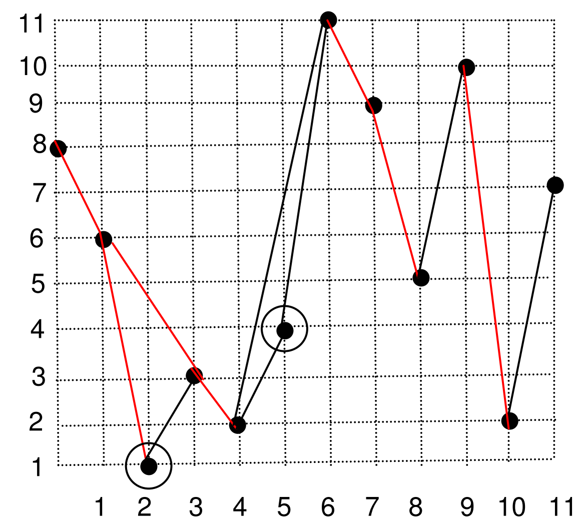
A permutation σ in S_n is minimal with *d* descents if:

1. σ has exactly *d* descents
2. σ does not contain π in S_k having *d* descents, $k<n$, as pattern

Minimal permutations with *d* descents: a first characterization

8 6 1 3 2 4 11 9 5 10 7 is not minimal with 6 descents

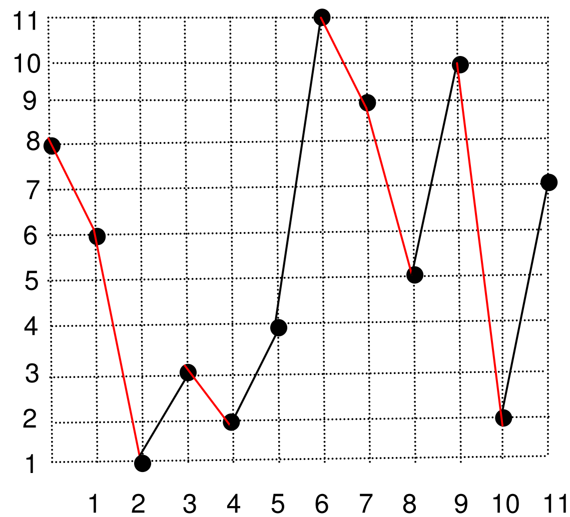


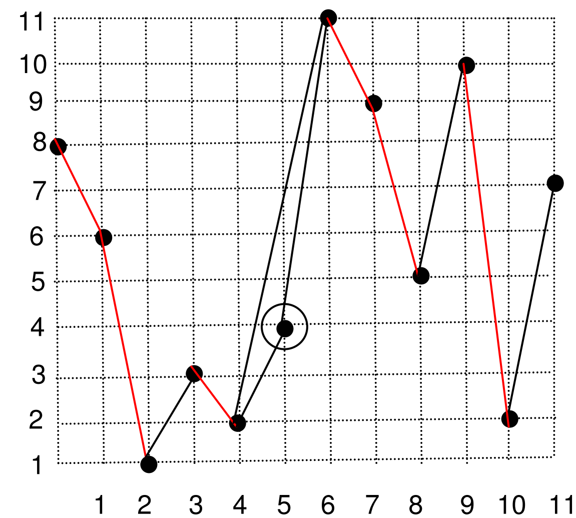


6 4 2 1 9 7 3 8 5 is minimal with 6 descents

Minimal permutations with *d* descents: a first characterization

σ minimal with *d* descents → no consecutive ascents in σ



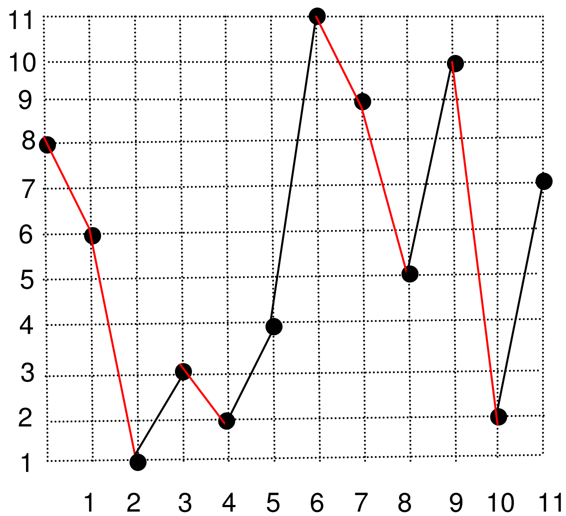


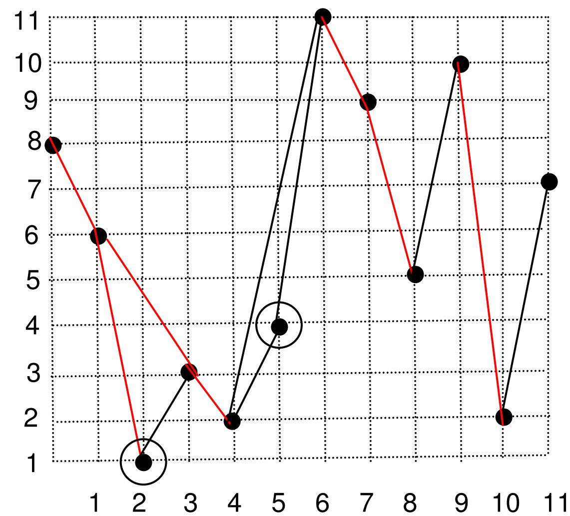
σ minimal with *d* descents → $d+1 \leq |\sigma| \leq 2d$

Minimal permutations with d descents: a first characterization

σ minimal with d descents → no consecutive ascents in σ

this condition is not sufficient

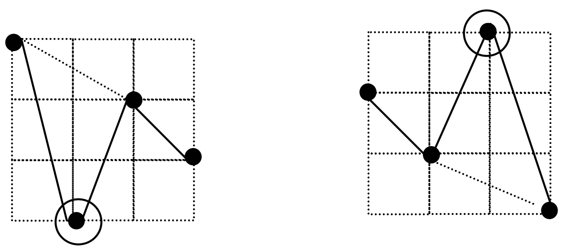


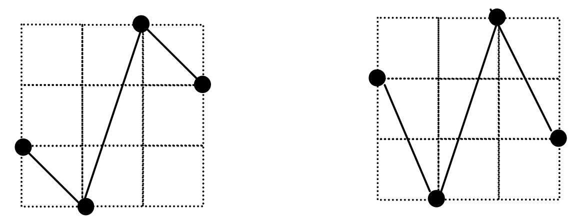


Minimal permutations with *d* descents: a local characterization

A permutation σ in S_n is minimal with *d* descents iff:

1. σ has exactly *d* descents
2. the four elements around each ascent of σ are ordered as *2143* or *3142*





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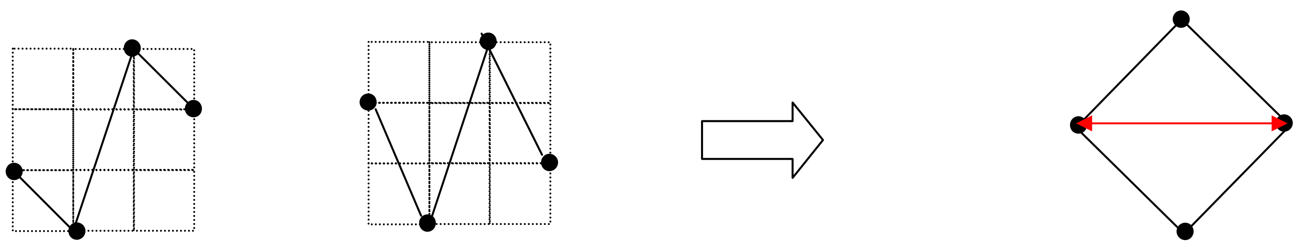

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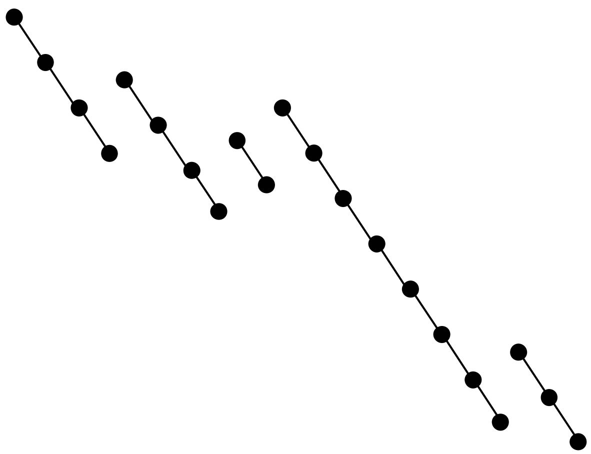

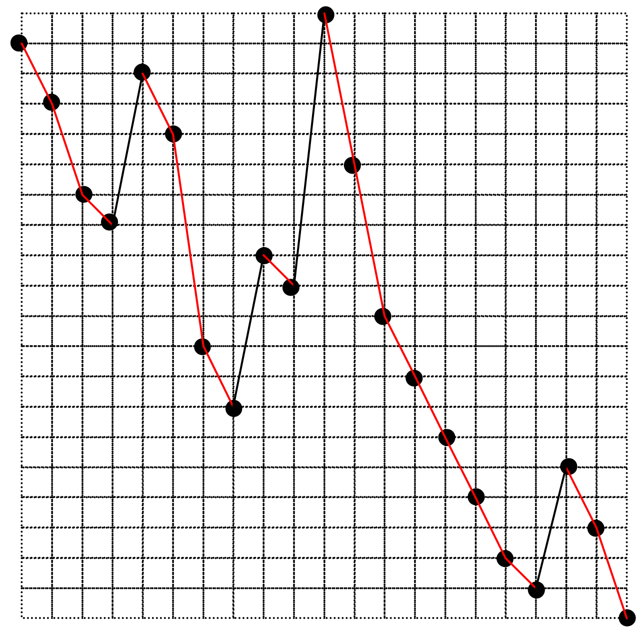

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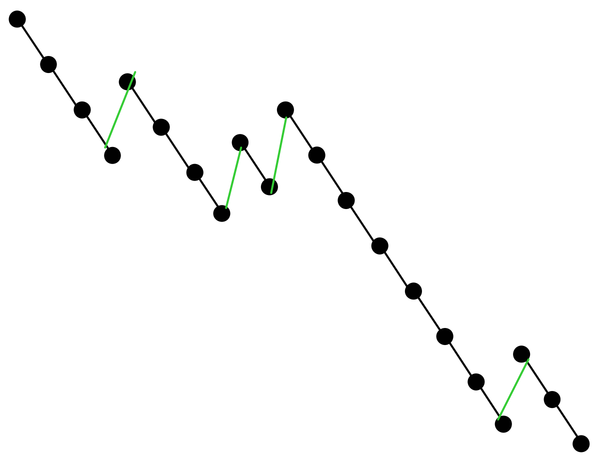
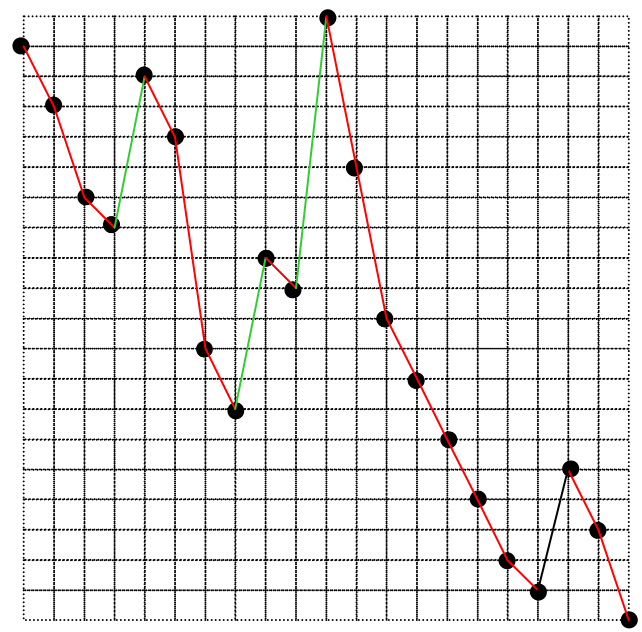
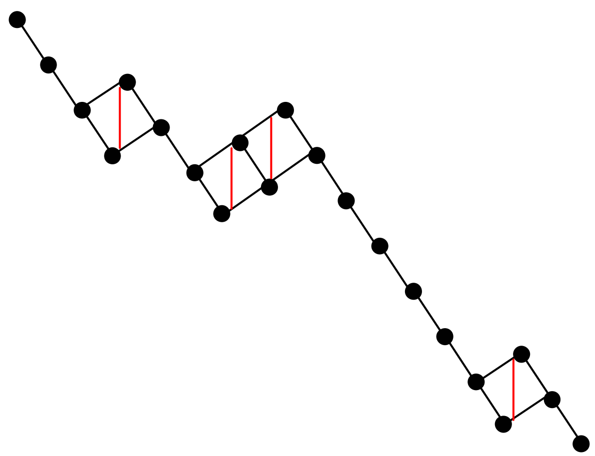
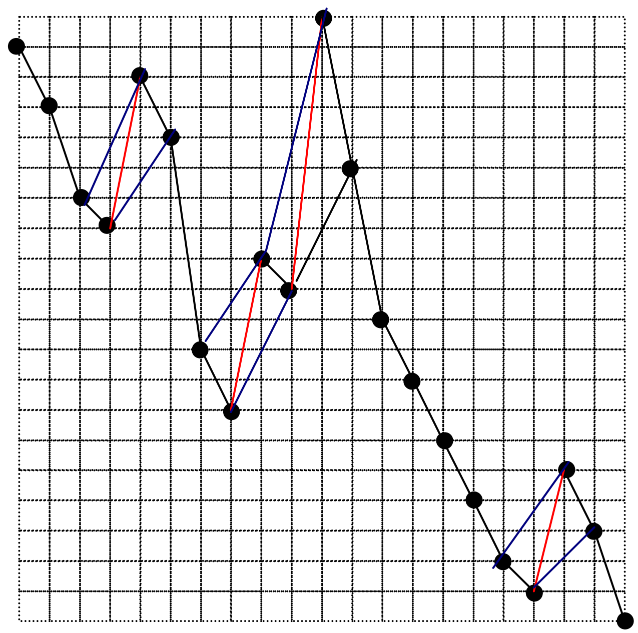


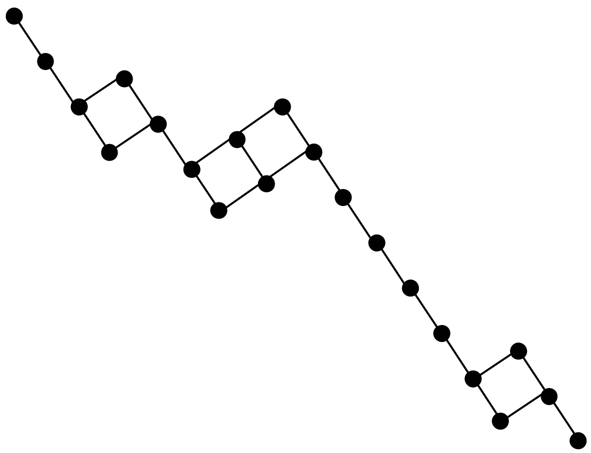


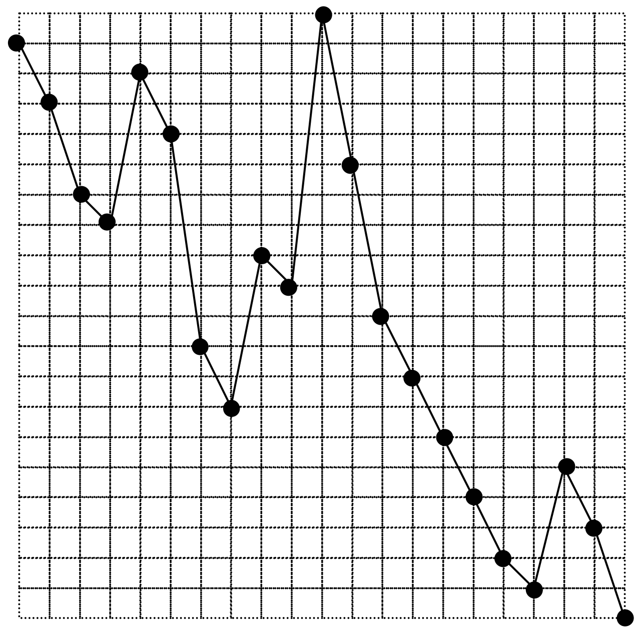
Minimal permutations with *d* descents: a local characterization



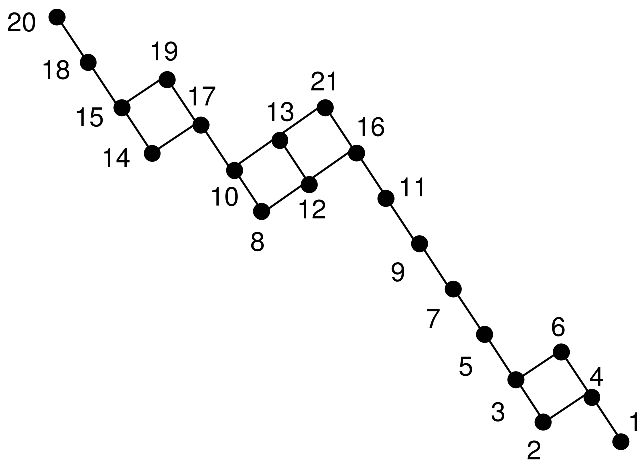


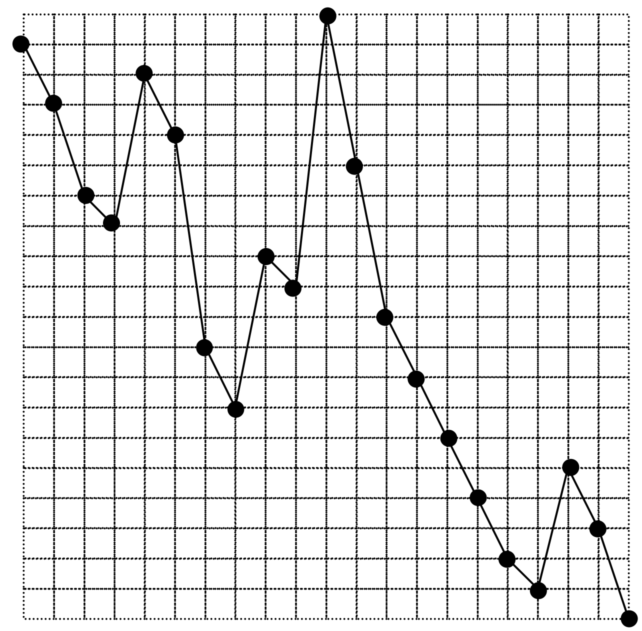
Minimal permutations with *d* descents: a local characterization





Minimal permutations with *d* descents: a local characterization





Permutation ↔ Authorized labelling of the posets

Words

Let $\Sigma_{2(n-1)} = \{w=b^l a b^{n-l-2} a^m b a^{m-l-2} : 1 \le l,m \le n-2\}$

bbbabbaabaaa is in Σ_{12}

bababbabbaaa is not in Σ_{12}

Sequence A0002890

Previous results

- the number of minimal permutation of lenght n $(=d+2)$ with n-2 descents is $2^n-n(n-1)-2$

- the number of permutations of leght n with n-2 descents is 2^n-n-1

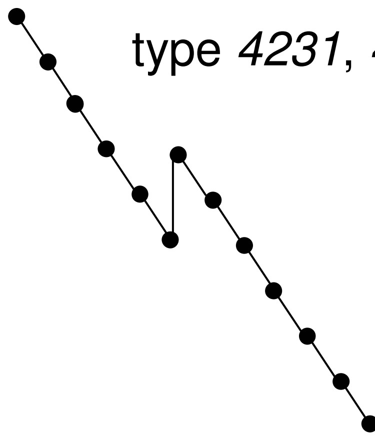
$$\Downarrow$$

the number of non minimal permutations of lenght n with 1 ascent is $(n-1)^2$

•Bouvel, Pergola, *Posets and Permutations in the duplication-loss model*, GASCOM08

Non minimal permutations

the number of non minimal permutations of lenght *n* with *1* ascent is $(n-1)^2$



type *4231*, *4132*, *3241*

2143, *3142* avoiding

Bijective enumeration: the algorithm

1. place a letter *b* at position *(n-1+i)*, *i* being he position of the ascent in the permutation,

2. let *k* be the number of consecutive elements greater than the *i-th element* of the permutation on its left, place a letter at position *(i-th element)+k*

3. complete the word by inserting *b* in the first possible *n* positions and *a* in the last possible *n-2* positions

Bijective enumeration: an example

0. *6 4 3 2 5 1* $\longrightarrow$ *x x x x x x x x x x*

Bijective enumeration: an example

0. *6 4 3 2 5 1* ⟶ *x x x x x x x x x x*

1. *6 4 3 2 5 1*

position *4*

Bijective enumeration: an example

0. $6\ 4\ 3\ 2\ 5\ 1 \longrightarrow x\ x\ x\ x\ x\ x\ x\ x\ x\ x$

1. $6\ 4\ 3\ 2\ 5\ 1 \xrightarrow{6-1+4} x\ x\ x\ x\ x\ x\ x\ x\ b\ x$

Bijective enumeration: an example

0.   *6 4 3 2 5 1* ⟶ *x x x x x x x x x x*

1.   *6 4 3 2 5 1* —*6-2+4*→ *x x x x x x x x b x*

2.   *6 4 3 2 5 1*

2 elements satisfying the property

Bijective enumeration: an example

0.	*6 4 3 2 5 1*	⟶		*x x x x x x x x x x*
1.	*6 4 3 2 5 1*	⟶	*6-2+4*	*x x x x x x x x b x*
2.	*6 4 3 2 5 1*	⟶	*2+2*	*x x x a x x x x b x*

Bijective enumeration: an example

0.	6 4 3 2 5 1	⟶	x x x x x x x x x x
1.	6 4 3 2 5 1	6-2+4 ⟶	x x x x x x x x b x
2.	6 4 3 2 5 1	2+2 ⟶	x x x a x x x x b x
3.	6 4 3 2 5 1	⟶	b b b a b a a a b a

Constructive enumeration by swapping: the algorithm

$\pi = n\ (n\text{-}1)\ldots(n\text{-}i\text{+}1)(n\text{-}i)\ldots 3\ 2\ 1$

1. $\pi_0 = n\ (n\text{-}1)\ldots\ldots(n\text{-}i\text{+}2)\ (n\text{-}i)\ (n\text{-}i\text{+}1)\ (n\text{-}i\text{-}1)\ldots 3\ 2\ 1$

2. $\pi_1 = n\ (n\text{-}1)\ldots(n\text{-}i\text{+}2)\ (n\text{-}i\text{-}1)\ (n\text{-}i\text{+}1)\ (n\text{-}i)\ldots..\ 1$

   ....

   $\pi_{n\text{-}i\text{-}1} = n\ (n\text{-}1)\ldots(n\text{-}i\text{+}2)\ 1\ (n\text{-}i\text{+}1)\ (n\text{-}i)\ldots..\ 2$

$1 \leq i \leq n\text{-}1$

3. $\pi_1 = n\ (n\text{-}1)\ldots(n\text{-}i\text{+}3)\ (n\text{-}i\text{+}1)\ (n\text{-}i)\ (n\text{-}i\text{+}2)\ (n\text{-}i\text{-}1)\ldots..\ 1$

   $\pi_2 = n\ (n\text{-}1)\ldots(n\text{-}i\text{+}2)(n\text{-}i\text{+}1)\ (n\text{-}i)\ (n\text{-}i\text{+}3)\ (n\text{-}i\text{-}1)\ldots..\ 1$

   ....

   $\pi_{i\text{-}1} = (n\text{-}1)\ldots(n\text{-}i\text{+}2)\ (n\text{-}i\text{+}1)\ (n\text{-}i)\ n\ (n\text{-}i\text{-}1)\ldots..\ 1$

Constructive enumeration by swapping : an example

i=1

π_0 = *56*4321

Constructive enumeration by swapping : an example

$i=1$

$\pi_0 = 564321$

$\pi_1 = 465321$

$\pi_2 = 364521$

$\pi_3 = 264351$

$\pi_4 = 164325$

Constructive enumeration by swapping : an example

i=1

$\pi_0 = 564321$

$\pi_1 = 465321$

$\pi_2 = 364521$

$\pi_3 = 264351$

$\pi_4 = 164325$

i=2

$\pi_0 = 645321$

Constructive enumeration by swapping : an example

i=1

$\pi_0 = 564321$

$\pi_1 = 465321$

$\pi_2 = 364521$

$\pi_3 = 264351$

$\pi_4 = 164325$

i=2

$\pi_0 = 645321$

$\pi_1 = 635421$

$\pi_2 = 625431$

$\pi_3 = 615432$

Constructive enumeration by swapping : an example

i=1

$\pi_0 = 564321$

$\pi_1 = 465321$

$\pi_2 = 364521$

$\pi_3 = 264351$

$\pi_4 = 164325$

i=2

$\pi_0 = 645321$

$\pi_1 = 635421$

$\pi_2 = 625431$

$\pi_3 = 615432$

$\pi_4 = 546321$

Constructive enumeration by swapping : an example

i=1

$\pi_0 = 564321$

$\pi_1 = 465321$

$\pi_2 = 364521$

$\pi_3 = 264351$

$\pi_4 = 164325$

i=2

$\pi_0 = 645321$

$\pi_1 = 635421$

$\pi_2 = 625431$

$\pi_3 = 615432$

$\pi_4 = 546321$

i=3

$\pi_0 = 653421$

Constructive enumeration by swapping : an example

i=1

π_0 = 564321

π_1 = 465321

π_2 = 364521

π_3 = 264351

π_4 = 164325

i=2

π_0 = 645321

π_1 = 635421

π_2 = 625431

π_3 = 615432

π_4 = 546321

i=3

π_0 = 653421

π_1 = 652431

π_2 = 651432

Constructive enumeration by swapping : an example

i=1

$\pi_0 = 564321$

$\pi_1 = 465321$

$\pi_2 = 364521$

$\pi_3 = 264351$

$\pi_4 = 164325$

i=2

$\pi_0 = 645321$

$\pi_1 = 635421$

$\pi_2 = 625431$

$\pi_3 = 615432$

$\pi_4 = 546321$

i=3

$\pi_0 = 653421$

$\pi_1 = 652431$

$\pi_2 = 651432$

$\pi_3 = 643521$

$\pi_4 = 543621$

Constructive enumeration by swapping : an example

i=1

$\pi_0 = 564321$

$\pi_1 = 465321$

$\pi_2 = 364521$

$\pi_3 = 264351$

$\pi_4 = 164325$

i=2

$\pi_0 = 645321$

$\pi_1 = 635421$

$\pi_2 = 625431$

$\pi_3 = 615432$

$\pi_4 = 546321$

i=3

$\pi_0 = 653421$

$\pi_1 = 652431$

$\pi_2 = 651432$

$\pi_3 = 643521$

$\pi_4 = 543621$

i=4

$\pi_0 = 654231$

$\pi_1 = 654132$

$\pi_2 = 653241$

$\pi_3 = 643251$

$\pi_4 = 543261$

i=5

$\pi_0 = 654312$

$\pi_1 = 654213$

$\pi_2 = 653214$

$\pi_3 = 643215$

$\pi_4 = 543216$

Constructive enumeration by swapping : the proof

- the obtained permutations are non minimal:

positions: *(i-1) i (i+1)(i+2)*

1. $\pi_0\,(i+2) < \pi_0\,(i) < \pi_0\,(i+1) < \pi_0\,(i-1)$ are of type 4231
2. $\pi_j\,(i+2) < \pi_j\,(i) < \pi_j\,(i+1) < \pi_j\,(i-1)$ are of type 4132
3. $\pi_j\,(i+2) < \pi_j\,(i) < \pi_j\,(i-1) < \pi_j\,(i+1)$ are of type 3241

Constructive enumeration by swapping : the proof

- all non minimal permutations are obtained

possible configuration

1. $\pi_0(i) < \pi_0(i-1) < \pi_0(i+2) < \pi_0(i+1)$ not admissible: type 2143
2. $\pi_0(i) < \pi_0(i+2) < \pi_0(i-1) < \pi_0(i+1)$ not admissible: type 3142
3. $\pi_0(i+2) < \pi_0(i) < \pi_0(i+1) < \pi_0(i-1)$ are of type 4231
4. $\pi_0(i) < \pi_0(i+2) < \pi_0(i+1) < \pi_0(i-1)$ are of type 4132
5. $\pi_0(i+2) < \pi_0(i) < \pi_0(i-1) < \pi_0(i+1)$ are of type 3241

Constructive enumeration by explosion: the algorithm

1. π of length n gives $(n+1)\ \pi$

2. π, $1 \leq i \leq n-1$, of length n gives $\pi'1$: $\pi'(i) = \pi(i)+1$

3. add the two permutations $n\ (n-2) \ldots 2\ 1\ (n+1)$ and $1\ (n+1)\ n \ldots 2$

Constructive enumeration by explosion: an example

12

Constructive enumeration by explosion: an example

12   *312*

Constructive enumeration by explosion: an example

231

12　　　　*312*

Constructive enumeration by explosion: an example

12 → *23*1 *132* / *3*12 *213*

	*23*1	*132*
12	*3*12	*213*

Constructive enumeration by explosion: an example

	231	*132*	*4231*	*4132*
12	*312*	*213*	*4312*	*4213*

Constructive enumeration by explosion: an example

			3421	*2431*	
	231	*132*	*4231*	*4132*	
12	*312*	*213*	*4312*	*4213*	*3241*

Constructive enumeration by explosion: an example

			3421	*2431*	*1432*
	231	*132*	*4231*	*4132*	*3214*
12	*312*	*213*	*4312*	*4213*	*3241*

Constructive enumeration by explosion: an example

			3421	*2431*	*1432*		*53421*	*52431*	*51432*
	231	*132*	*4231*	*4132*	*3214*		*54231*	*54132*	*53214*
12	*312*	*213*	*4312*	*4213*	*3241*		*54312*	*54213*	*53241*

Constructive enumeration by explosion: an example

						45321	35421	25431	
			3421	2431	1432	53421	52431	51432	
	231	132	4231	4132	3214	54231	54132	53214	43251
12	312	213	4312	4213	3241	54312	54213	53241	43521

Constructive enumeration by explosion: an example

						45321	35421	25431	15432
			3421	2431	1432	53421	52431	51432	43215
	231	132	4231	4132	3214	54231	54132	53214	43251
12	312	213	4312	4213	3241	54312	54213	53241	43521

Statistics

n / π_1	3	4	5	6	7	8
1	1	1	1	1	1	1
2	2	1	1	1	1	1
3	1	3	1	1	1	1
4		4	4	1	1	1
5			9	5	1	1
6				16	6	1
7					25	7
8						36

1. the permutation with the first entry equal to *1,… , (n-2)* are *1* for each *n*
2. the permutation with the first entry equal to *(n-1)* are *(n-1)* for each *n*
3. the permutation with the first entry equal to *n* are $(n-2)^2$ for each *n*

Statistics

complement permutation: $\pi(i)^c = n+1- \pi(i)$

reverse permutation: $\pi(i)^r = \pi(n+1-i)$

2 6 5 4 3 1 → 1 3 4 5 6 2 → 6 4 3 2 1 5

Let $\pi = \pi(1) \ldots \pi(n)$ be a non minimal permutation then $((\pi)^c)^r$ is a non minimal permutation such that if $\pi(1) = l$ then $\pi(n) = n+1-l$

Statistics

π(1) \ n	3	4	5	6	7	8
1	1	1	1	1	1	1
2	2	1	1	1	1	1
3	1	3	1	1	1	1
4		4	4	1	1	1
5			9	5	1	1
6				16	6	1
7					25	7
8						36

π(n) \ n	3	4	5	6	7	8
1	1	4	9	16	25	36
2	2	3	4	5	6	7
3	1	1	1	1	1	1
4		1	1	1	1	1
5			1	1	1	1
6				1	1	1
7					1	1
8						1

About non minimal permutations