Packing Set Partitions

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Outline

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- I'm introduced to packing in Iceland and am encouraged to consider packing for set partitions.
- Today's talk/plea for help.

Set Partition Definition

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Example

$137/25/46 \vdash [7]$

Canonical Words

To each set partition is associated a canonical word $a_1 a_2 \ldots a_n$ where $a_i = j$ if $i \in B_j$.

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$137/25/46$ *corresponds to* 1213231.

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Given any word $w \in [k]^n$ we may canonize w by replacing all occurrences of the first letter of w by 1, all occurrences of the next occurring letter by 2, etc.

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From now on we will refer to these canonical words as partitions.

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1. *Traditional:*
   there is a subsequence of σ of length k whose canonization is π.
2. *Restricted:*
   there is a subsequence of σ of length k that is order isomorphic to π.

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And 131*, as in* 1213231*, is a copy of 121 in both the traditional and restricted sense.*

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Then 212*, as in* 1213231 *is a copy of 121 in the traditional sense, but not in the restricted sense.*

And 131*, as in* 1213231*, is a copy of 121 in both the traditional and restricted sense.*

The traditional pattern containment definition allows for the blocks of the copy to be in a different order than the blocks in the pattern. The restricted definition does not.

Packing Definitions

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d(E,\sigma) &= \frac{\nu(E,\sigma)}{\binom{n}{m}} \\
d(E,k,n) &= \frac{\mu(E,k,n)}{\binom{n}{m}}
\end{aligned}$$

Density

Theorem

For $n \geq m$ *we have that* $d(E, k, n-1) \geq d(E, k, n)$ *and* $d(E, k, n) \geq d(E, k-1, n)$. $\square$

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Definition

Let the packing density of the set $E \subseteq \Pi_m$ be

$$\delta(E) = \lim_{n\to\infty} d(E, n, n) = \lim_{n\to\infty} \lim_{k\to\infty} d(E, k, n).$$

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Restricted Results

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- All of the results of Price [3] and Albert, Atkinson, Handley, Holton, and Stromquist [1] about layered permutations apply.
- $\delta(121) = \frac{2\sqrt{3}-3}{2} \approx 0.2321$

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- For example $\pi = 112223334444444$ is monotone layered, but $\pi = 112333$ is not monotone layered.
- If *E* consists entirely of monotone layered partitions then there exists some σ, which is also monotone layered, such that $\nu(E, \sigma) = \mu(E, n, n)$.

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- If E consists entirely of monotone layered partitions then there exists some σ, which is also monotone layered, such that $\nu(E, \sigma) = \mu(E, n, n)$.
- Thus, all results for layered permutations that are monotone apply.

Monotone Layered Theorem

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Theorem

Given a monotone increasing layered pattern $\pi = 1 \dots 12 \dots 2 \dots k \dots k$ then among all partitions with the same block structure as σ, the layered monotone increasing one is a maximizer.

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Let $\hat{j}$ be the letter corresponding to *j* in $\hat{\sigma}$.
By induction on the length of σ the number of copies of π in σ that do not involve this *j* is no more than the number of copies in $\hat{\sigma}$ that do not involve $\hat{j}$.

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By induction on the length of σ the number of copies of π in σ that do not involve this *j* is no more than the number of copies in $\hat{\sigma}$ that do not involve $\hat{j}$.
Now consider the number of copies of π in σ that did involve this *j*. In any such copy the *j*s must correspond to the *k*s of π. By inducting on *k* we can say that number of copies has not decreased in this case as well. □

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The partition of $[n]$ *that maximizes the number of copies of* 121 *is* $\underbrace{121212\ldots}_{n}$

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Theorem

$\delta(121) \leq \frac{1}{2}$. $\square$

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- Can we solve the layered problem?
- Can it be shown that 12121 . . . is the best possible for packing 121?
- Can we at least get a better upper bound for 121?
- Anything that might translate from the permutation case to the set partition case.

THANK YOU

M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton, W. Stromquist, On packing densities of permutations, Electron. J. Combin. 9 (1) (2002) Research Paper 5, 20 pp. (electronic).

A. Burstein, P. Hästö, T. Mansour, Packing patterns into words, Electron. J. Combin. 9 (2) (2002/03) Research paper 20, 13 pp. (electronic), permutation patterns (Otago, 2003).

A. Price, Packing densities of layered patterns, Ph.D. thesis, University of Pennsylvania, Philadelphia, PA, 1997.