The pattern 31-2 in alternating permutations

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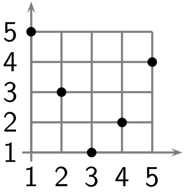
Permutation Patterns 2009, Firenze

Introduction

Definition

An occurrence of the pattern 31-2 in $\sigma \in \mathfrak{S}_n$ is a triple $i < i+1 < j$ such that $\sigma(i+1) < \sigma(j) < \sigma(i)$.

Example



This pattern appears in various context:

- q-analog of Eulerian numbers [Williams]
- Laguerre histories (weighted Motzkin paths) [Corteel]
- permutation tableaux [Steingrìmsson, Williams, Corteel, Nadeau]
- moments of rescaled Al-Salam-Chihara polynomials [Kasraoui, Stanton, Zeng]

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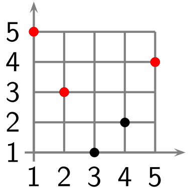

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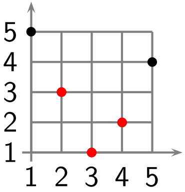

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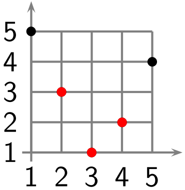

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- q-analog of Eulerian numbers [Williams]

$$E_{kn} = \#\{\sigma \in \mathfrak{S}_n \mid k \text{ ascents } \} \qquad E_{kn}(q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ k \text{ ascents}}} q^{31\text{-}2(\sigma)}$$

- Laguerre histories (weighted Motzkin paths) [Corteel]

Introduction

A permutation σ is alternating if $\sigma(1) > \sigma(2) < \sigma(3) > \dots$
Let E_n be the number of alternating permutations of size n.
We have [Euler]:

$$\sum_{k=0}^{n} (-1)^k E_{kn} = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{asc(\sigma)} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n & \text{if } n \text{ is odd.} \end{cases}$$

Let $\mathfrak{D}_n$ be the set of derangements of size n.
Let $wex(\sigma)$ be the number of weak exceedances:

$$wex(\sigma) = \#\{ i | \quad i \leq \sigma(i) \quad \}$$

We have [Roselle]:

$$\sum_{\sigma \in \mathfrak{D}_n} (-1)^{wex(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} E_n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Outline

- Refinements of Euler and Roselle identities with the 31-2 pattern
  - Laguerre histories, Françon-Viennot bijection, first identity
  - The second identity
- Closed formulas for $E_n(q)$
  - A decomposition of lattice paths
  - Enumeration of lattice paths
- Conclusion

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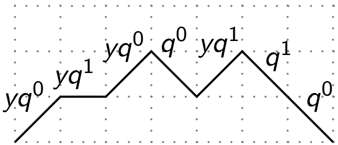
A *Laguerre history* is a weighted Motzkin path such that:

- the weight of a step $\nearrow$ starting at height h is yq^i with $0 \leq i \leq h$,
- the weight of a step $\rightarrow$ starting at height h is either yq^i or q^i, with $0 \leq i \leq h$,
- the weight of a step $\searrow$ starting at height h is q^i, with $0 \leq i \leq h$.

Example



Proposition (Françon-Viennot, Corteel)

- *There is a bijection between $\mathfrak{S}_n$ and the set of Laguerre histories of $n-1$ steps.*
- *The image of $\sigma \in \mathfrak{S}_n$ has total weight $y^{asc(\sigma)} q^{31\text{-}2(\sigma)}$ (the total weight is the product of the weights of each step).*

Let $\sigma \in \mathfrak{S}_n$.
We use the convention that $\sigma(0) = 0$ and $\sigma(n+1) = 0$.

Let $k = \sigma(j)$, the kth step in the Laguerre history is:

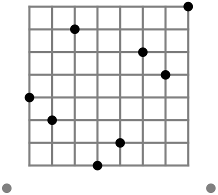
- a step $\nearrow$ if $\sigma(j-1) > \sigma(j) < \sigma(j+1)$
- a step $\searrow$ if $\sigma(j-1) < \sigma(j) > \sigma(j+1)$
- a step $\rightarrow$ if $\sigma(j-1) > \sigma(j) > \sigma(j+1)$ or $\sigma(j-1) < \sigma(j) < \sigma(j+1)$

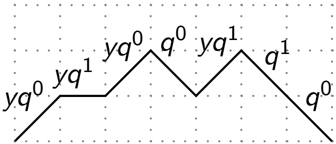
the weight of the kth step is $y^\delta q^u$ where

- $\delta = 1$ if j is an ascent and 0 otherwise,
- u is the number of triples $(x, x+1, j)$ which are occurrences of 31-2.

Example

$\sigma = 43712658$





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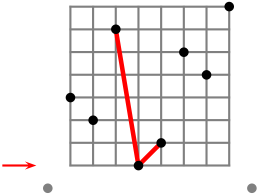

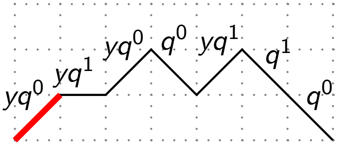

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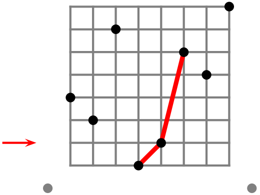

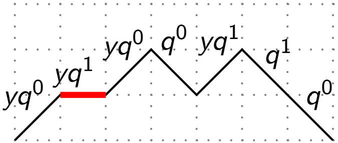

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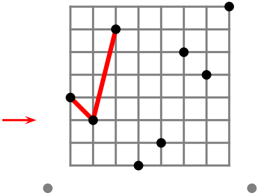

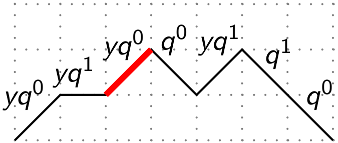

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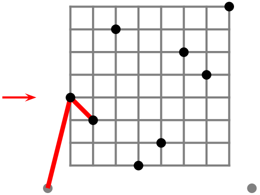

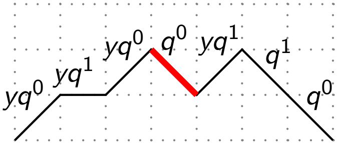

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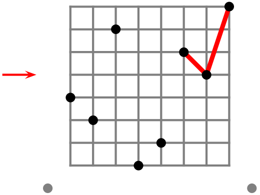

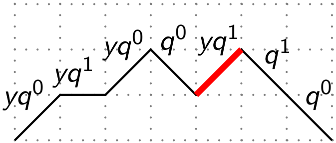

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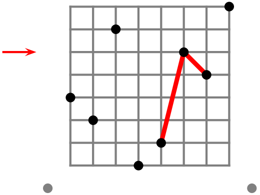

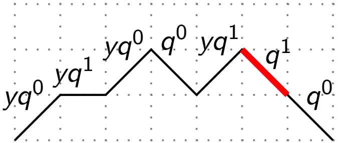

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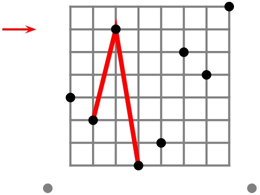

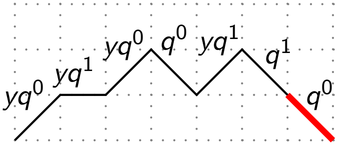

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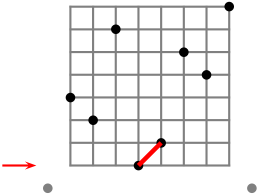

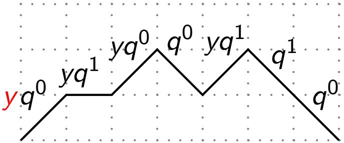

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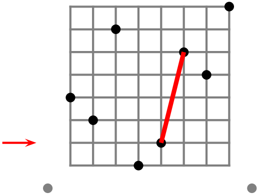

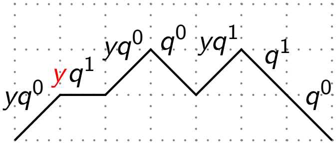

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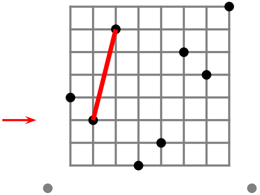

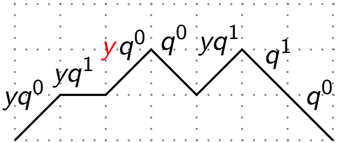

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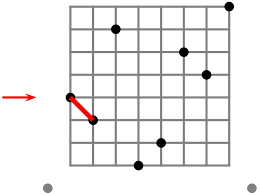

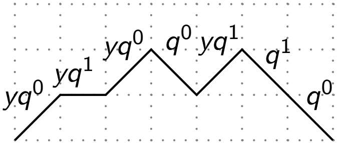

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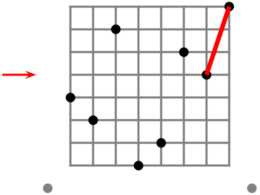

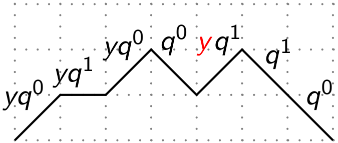

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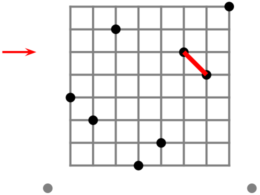

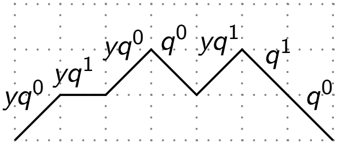

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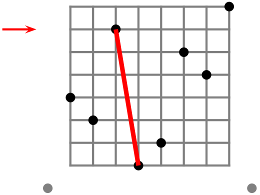

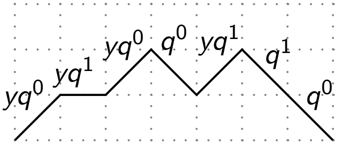

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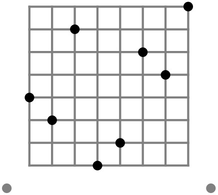

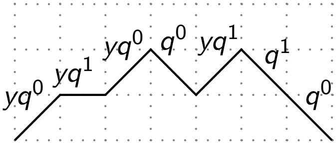

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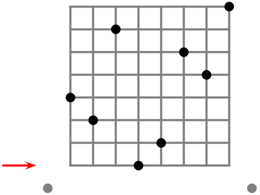

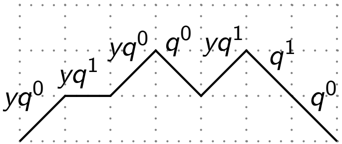

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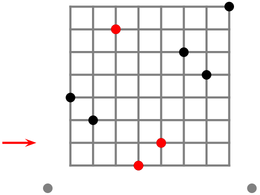

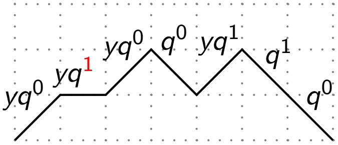

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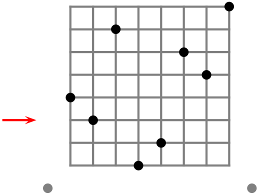

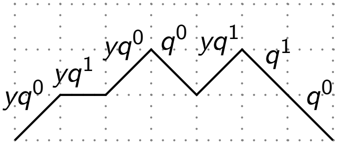

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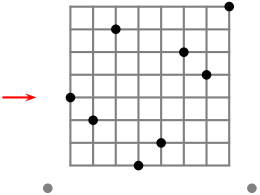

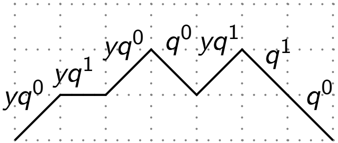

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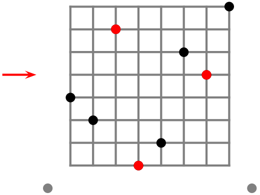

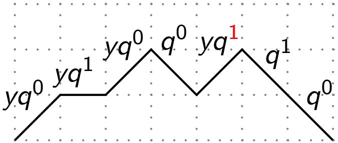

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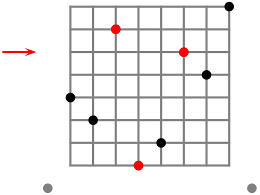

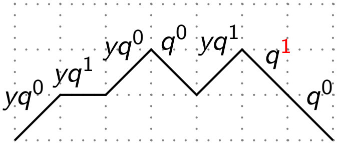

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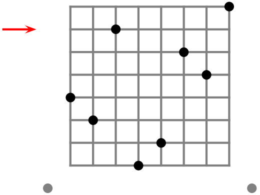
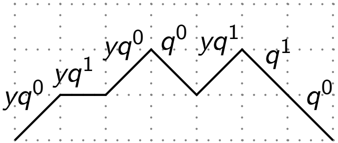




Proposition

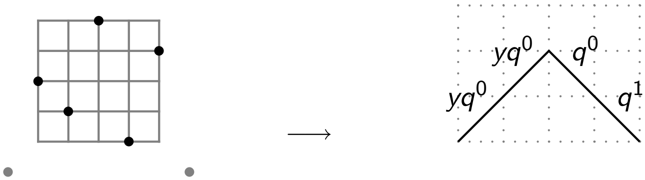
Via the Françon-Viennot bijection, The alternating permutations of odd size correspond to Laguerre histories without step $\rightarrow$.

Proof.

The alternating permutations of odd size are exactly the ones with no double ascent, no double descent.

Via the Françon-Viennot bijection, double ascents and double descents correspond to the horizontal steps. □

Example



Let

$$E_n(q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ alternating}} q^{31\text{-}2(\sigma)}.$$

Theorem

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{asc(\sigma)} q^{31\text{-}2(\sigma)} = \begin{cases} 0 & \textit{if } n \textit{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n(q) & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof.

$\sum y^{asc(\sigma)} q^{31\text{-}2(\sigma)}$ is the sum of weights of Laguerre histories of length $n-1$.

When $y=-1$, the possible weights on the horizontal steps are q^i or $-q^i$. So they cancel each other and we can restrict the sum to Laguerre histories without step $\rightarrow$ (which correspond to alternating permutations of odd size, thanks to the previous proposition). □

The "dual" identity

Let $\mathfrak{D}_n$ be the set of derangements.
Let $wex(\sigma)$ be the number of weak exceedances:

$$wex(\sigma) = \#\{i | \quad i \leq \sigma(i) \quad \}$$

Let $cr(\sigma)$ be the number of crossings:

$$cr(\sigma) = \#\Big\{(i,j) \Big| \quad i < j \leq \sigma(i) < \sigma(j) \quad \text{or} \quad \sigma(i) < \sigma(j) < i < j \quad \Big\}$$

Theorem

$$\sum_{\sigma \in \mathfrak{D}_n} \left(-\frac{1}{q}\right)^{wex(\sigma)} q^{cr(\sigma)} = \begin{cases} \left(-\frac{1}{q}\right)^{\frac{n}{2}} E_n(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Sketch of proof

We can use another version of the Françon-Viennot bijection, with the convention that $\sigma(0) = 0$ and $\sigma(n+1) = n+1$.

Now permutations of size n are in bijection with Laguerre histories of n steps, and no steps starting at height h with weight q^h (*restricted* Laguerre history).

Now the alternating permutations of *even* size are exactly the ones with no double ascent and no double descent.

The Foata-Zeilberger bijection sends derangements to restricted Laguerre histories with no step $\rightarrow$ with weight yq^0.
In this case, setting $y = -\frac{1}{q}$ cancels the weights on the horizontal steps.

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- Conclusion

The generating function for pattern 31-2 in permutations is

$$\sum_{\sigma\in\mathfrak{S}_n} q^{31\text{-}2(\sigma)} = \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\left(\binom{2n}{n-k}-\binom{2n}{n-k-2}\right)\sum_{i=0}^{k} q^{i(k+1-i)}$$

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Theorem

$$E_{2n+1}(q) = \frac{1}{(1-q)^{2n+1}}\sum_{k=0}^{n}\left(\binom{2n+1}{n-k}-\binom{2n+1}{n-k-1}\right)\sum_{i=0}^{2k+1}(-1)^{i+k}q^{i(2k+2-i)}.$$

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Sketch of proof for $E_{2n}(q)$

Inspired from [Penaud, Bij. proof of Touchard-Riordan formula].

$E_{2n}(q)$ is the sum of weights of Dyck paths of length $2n$ such that:

- the weight of a step between height h and $h+1$ is q^i with $0 \leq i \leq h$.

So $E_{2n}(q)$ is the sum of weights of Dyck paths of length $2n$, such that:

- the weight of a step between height h and $h+1$ is $\sum_{i=0}^{h} q^i$.

So $(1-q)^{2n}E_{2n}(q)$ is the sum of weights of weighted Dyck paths of length $2n$, such that:

- the weight of a step between height h and $h+1$ is $1-q^h$. (the weight of each step is multiplied by $1-q$).

Sketch of proof for $E_{2n}(q)$

$$(1-q)^{2n}E_{2n}(q) = \sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)\left(\sum_{i=0}^{2k}(-1)^{i+k}q^{i(2k-i)+k}\right).$$

Proposition

The number of Dyck prefixes of length $2n$ and final height $2k$ is $\binom{2n}{n-k}-\binom{2n}{n-k-1}$.

Proof. Recurrence or generating function □

Proposition

The sum of weights of weighted Dyck paths of length $2k$, with

- *weights 1 or $-q^h$ for a step between height h and $h+1$,*
- *no two steps $\nearrow\searrow$ both with weights 1,*

is $\sum_{i=0}^{2k}(-1)^{i+k}q^{i(2k-i)+k}$.

Proof. Continued fractions and basic hypergeometric series (non-combinatorial, hence omitted) □

So it remains to prove:
There is a weight-preserving bijection between

- Dyck path P of length $2n$ with weights 1 or $-q^h$ for any step starting at height h
- couples (G, H) such that for some $0 \leq k \leq n$
  - G is a Dyck prefix of length $2n$ and final height height $2k$,
  - H is a Dyck path of length $2k$, with
    - weights 1 or $-q^h$ for a step between height h and $h+1$,
    - no two steps $\nearrow\searrow$ both with weights 1,

Proof.

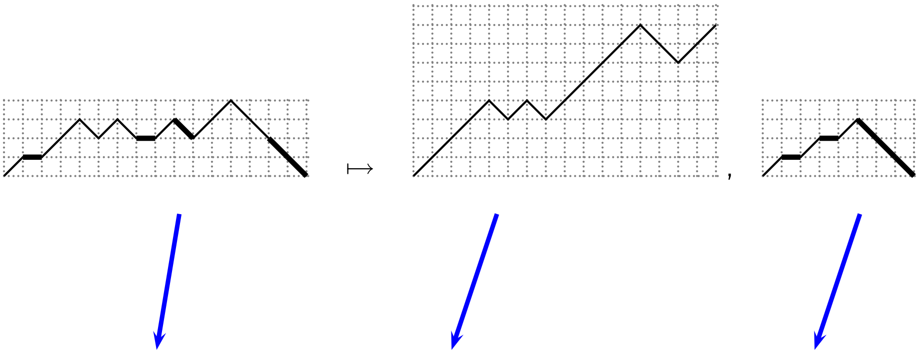
Consider the maximal factors in P which are Dyck paths and have no step with weight q^i.

- To obtain G, tranform into a $\nearrow$ any step of these maximal factors.
- To obtain H, remove these maximal factors.

□

Example

The steps ╱ and ╲ have weight 1.
The steps **╱** and **╲** have weight $-q^h$.



$$(1-q)^{2n}E_{2n}(q) \qquad \binom{2n}{n-k}-\binom{2n}{n-k-1} \qquad \sum_{i=0}^{2k}(-1)^{i+k}q^{i(2k-i)+k}$$

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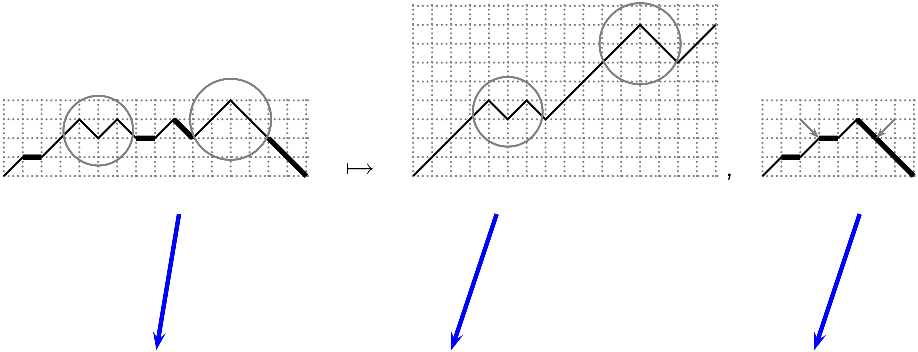

$$(1-q)^{2n} E_{2n}(q) \qquad \binom{2n}{n-k} - \binom{2n}{n-k-1} \qquad \sum_{i=0}^{2k} (-1)^{i+k} q^{i(2k-i)+k}$$

Concluding remarks

- There is similarly a half-combinatorial proof for $E_{2n+1}(q)$.
- We have no completely combinatorial proofs.
- They are very similar to the Touchard-Riordan formula, which counts the *crossings* in *perfect matchings*:

$$\sum_{m \in \mathfrak{M}_{2n}} q^{\mathrm{cr}(I)} = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) q^{\frac{k(k+1)}{2}}$$

$n = 2 \quad \longrightarrow$  $\longrightarrow \quad 2 + q$

[Penaud, Bijective proof of a Touchard-Riordan formula]

- Further work: the pattern 31-2 in Dumont permutations ?
  (I thank Alex Burstein !)