The Möbius function of the permutation pattern poset

Einar Steingrímsson
Reykjavík University

Joint work (in progress) with Bridget Tenner (and Eric Babson)

Theorem (M. Friedman): There is no free lunch.

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To compute the Möbius function for the pattern poset

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Wilf (2002): Should be done

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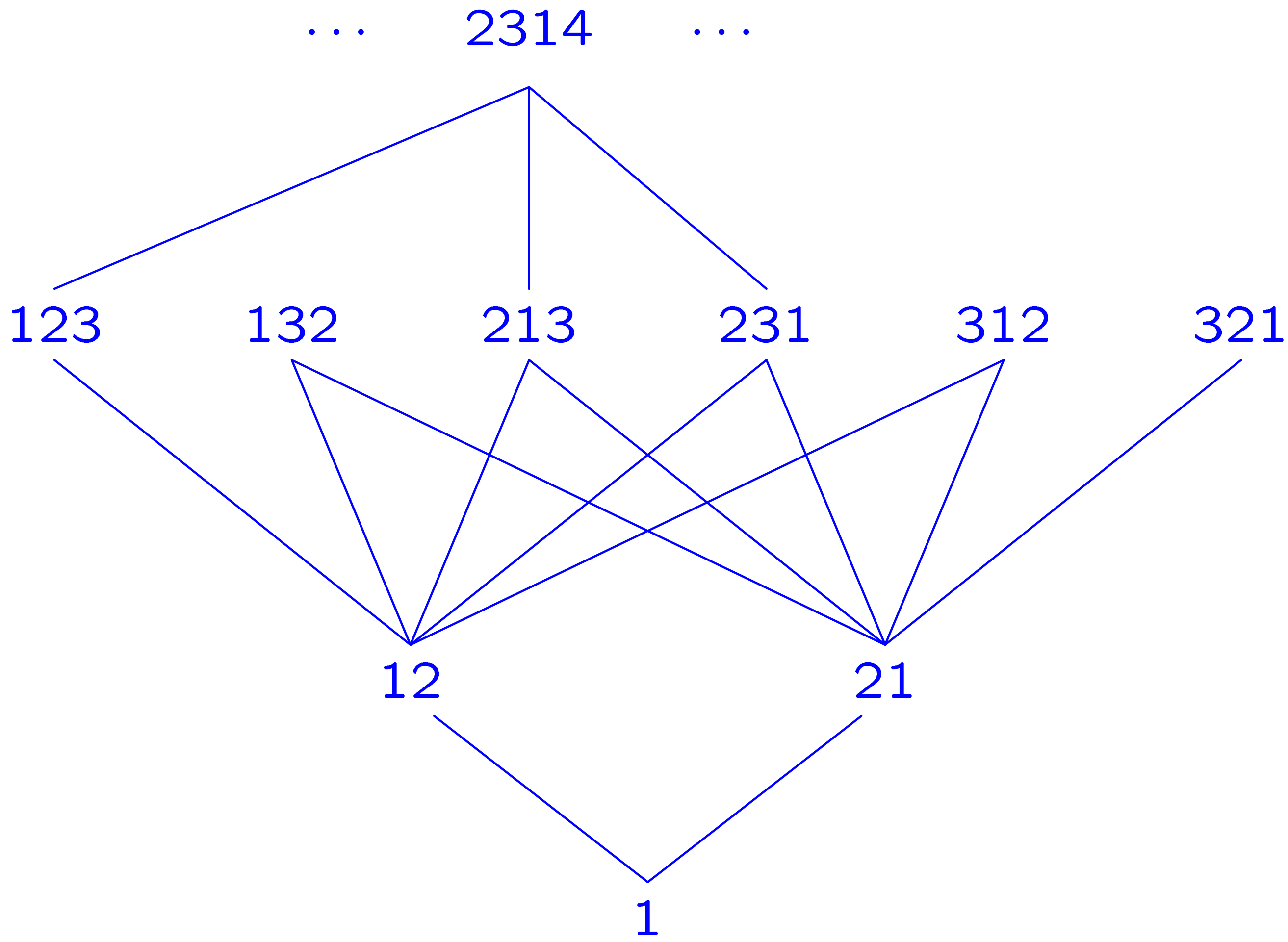
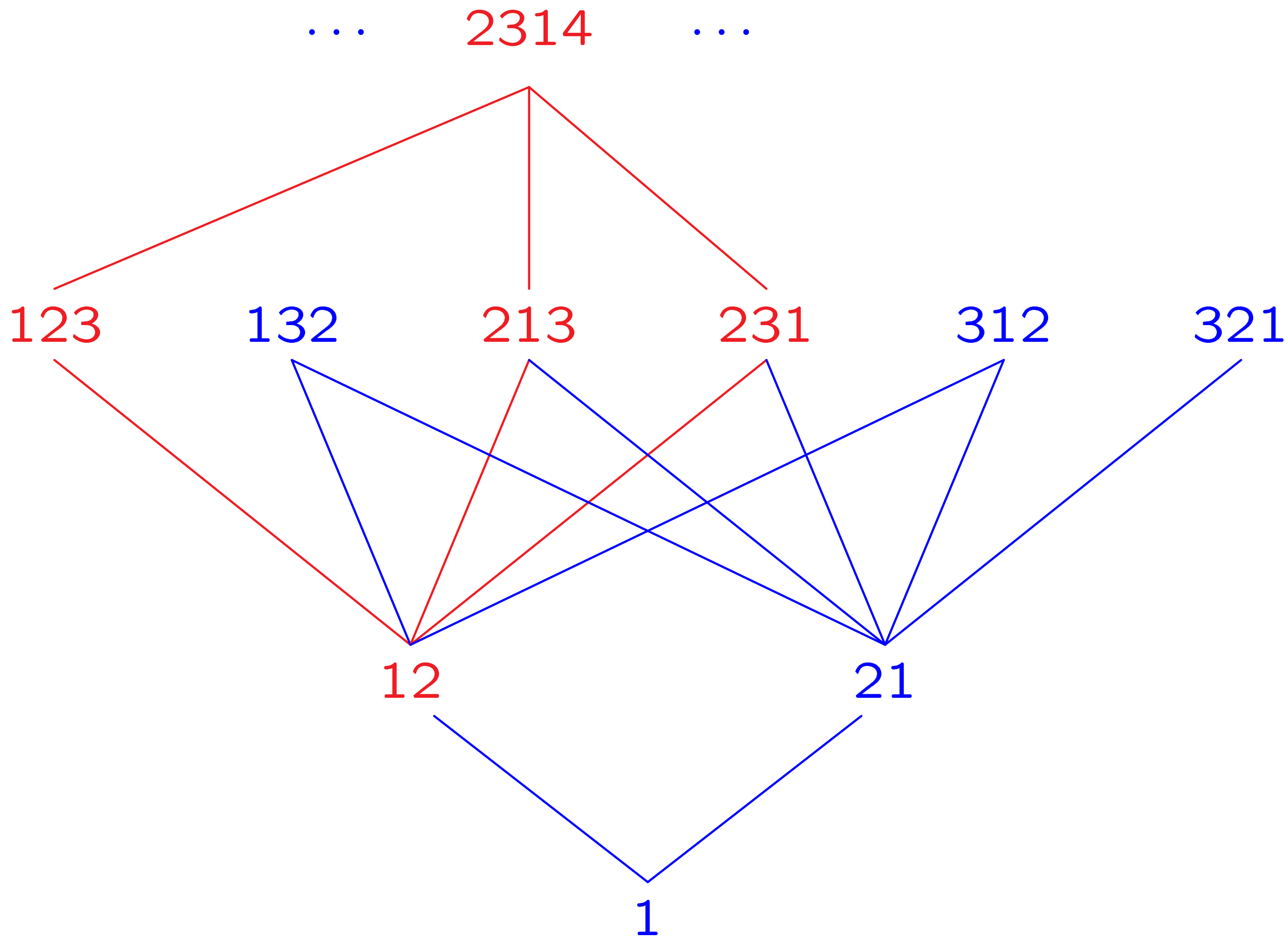
Sagan-Vatter (2005): Solution for layered permutations



The bottom of the poset $\mathcal{P}$



The interval [12, 2314]

The interval $[12, 2134]$:

2134

12

The interval $[12, 2134]$:

2134

12

The interval $[12, 2134]$:

2134

213

12

The interval $[12, 2134]$:

2134

213 214 234 134

12

The interval $[12, 2134]$:

2134

213 214 234 134

12

The interval $[12, 2134]$:

2134

213 234 134

12

The interval $[12, 2134]$:

2134

213 234 134

12

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2134

213 234 134

12

The interval $[12, 2134]$:

2134

213 234 134

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213 234 134

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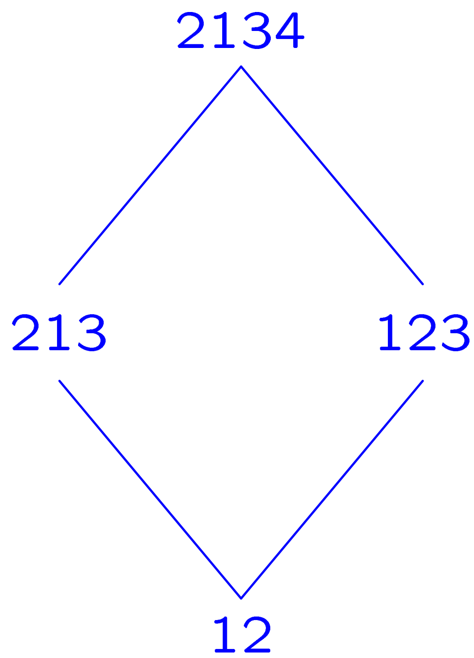
The interval $[12, 2134]$:

2134

213 123

12

The interval $[12, 2134]$:



4312

12

4312

312 412 432 431

12

4312

312 412 432 431

12

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312 432 431

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4312

312 ~~432~~ ~~431~~

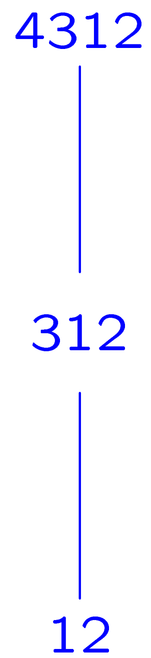
12

4312

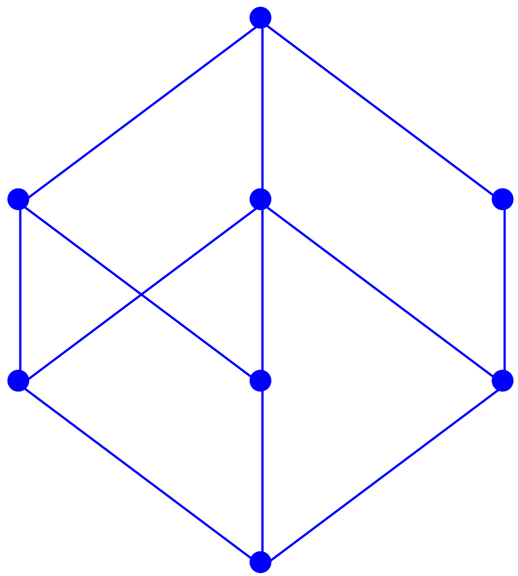
312 ~~432~~ ~~431~~

don't contain 12

12


4312
312
12

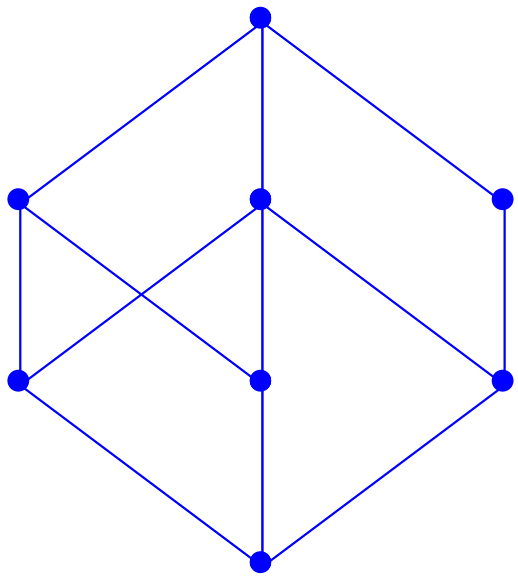
Computing the Möbius function



Def: The Möbius function of P is determined by $\mu(x, x) = 1$ and

$$\sum_{x \leqslant t \leqslant y} \mu(x, t) = 0 \quad \text{if } x < y$$

Computing the Möbius function



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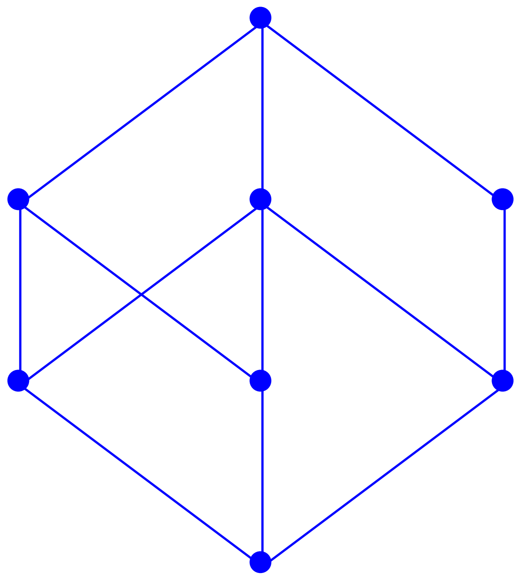

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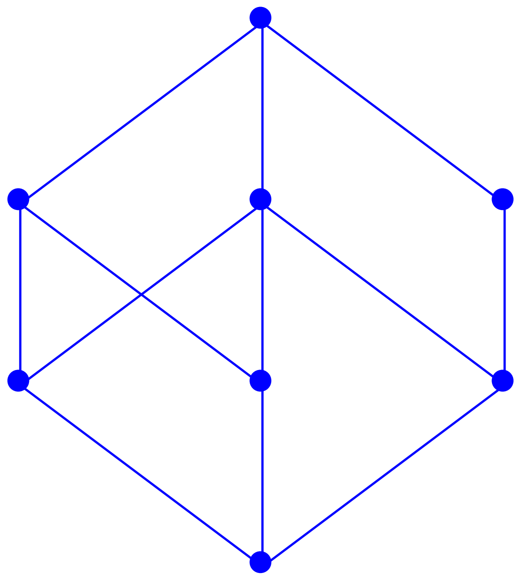

1

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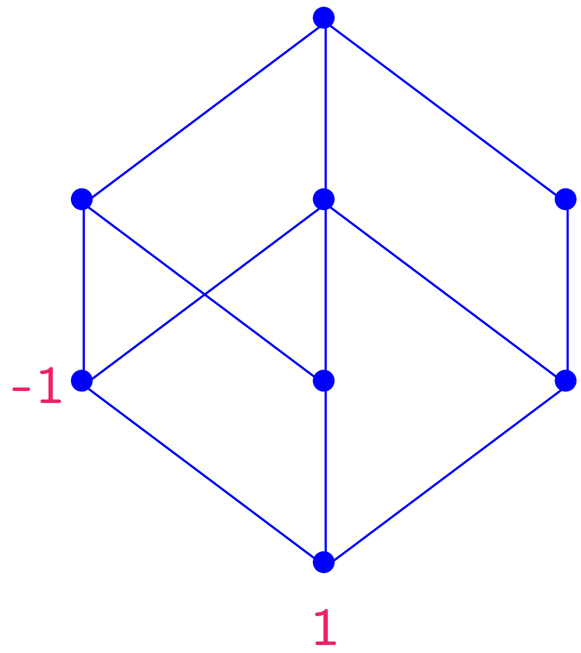

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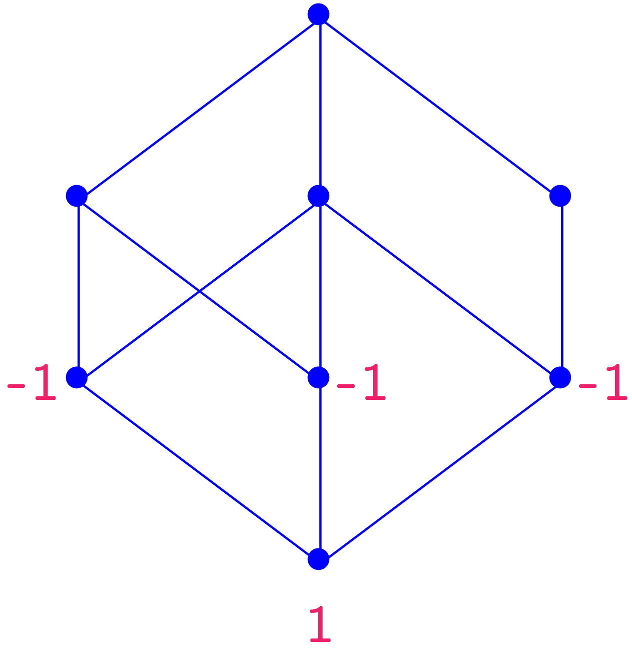

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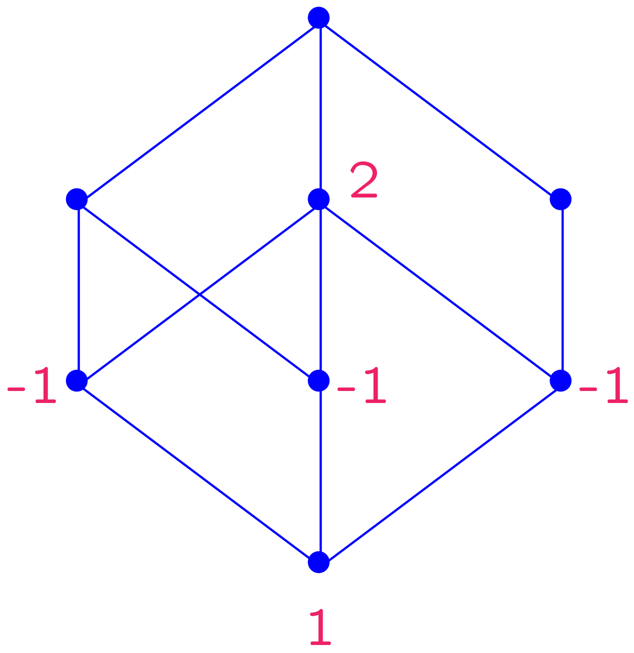

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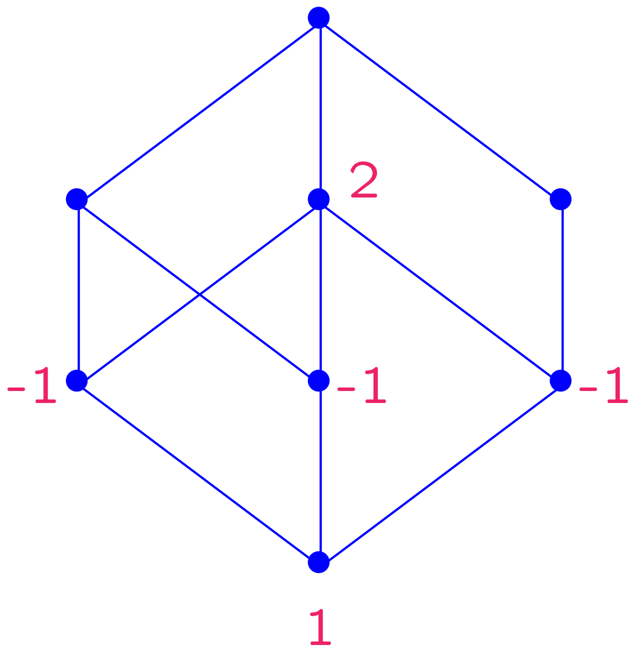
Computing the Möbius function

$\mu(x, y) = 2$



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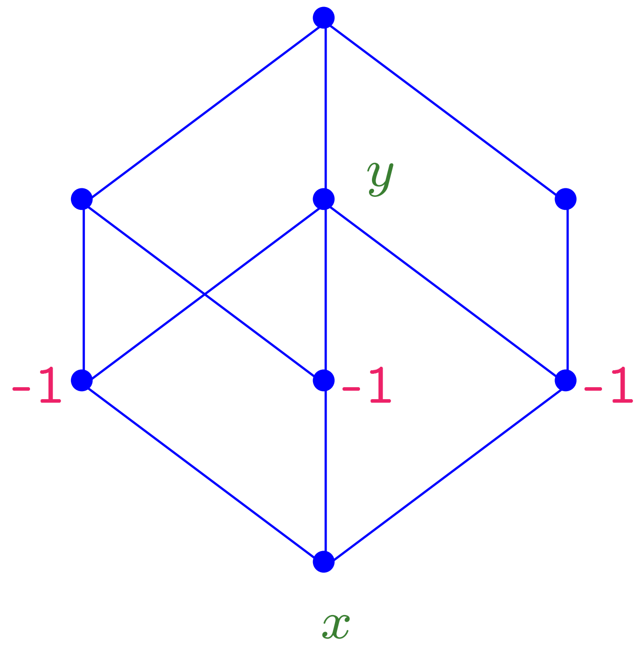

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Computing the Möbius function

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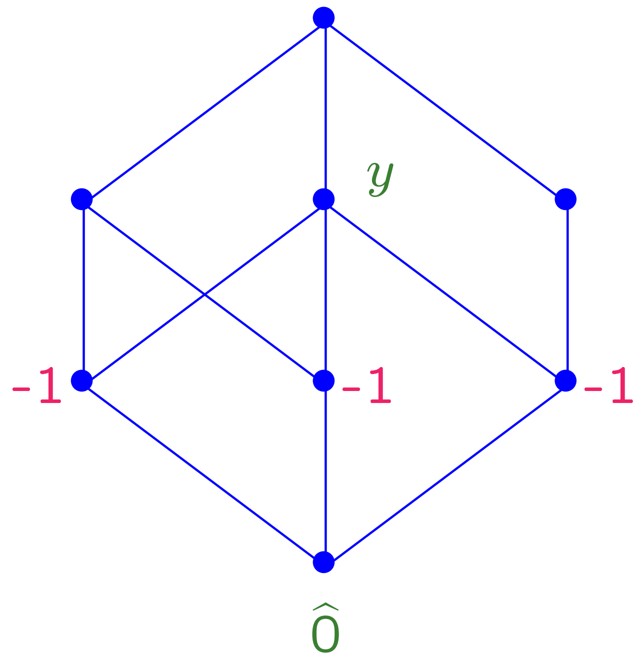

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Computing the Möbius function

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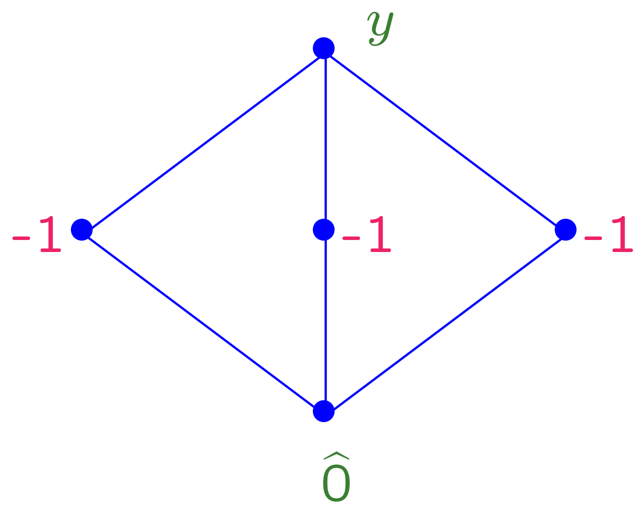

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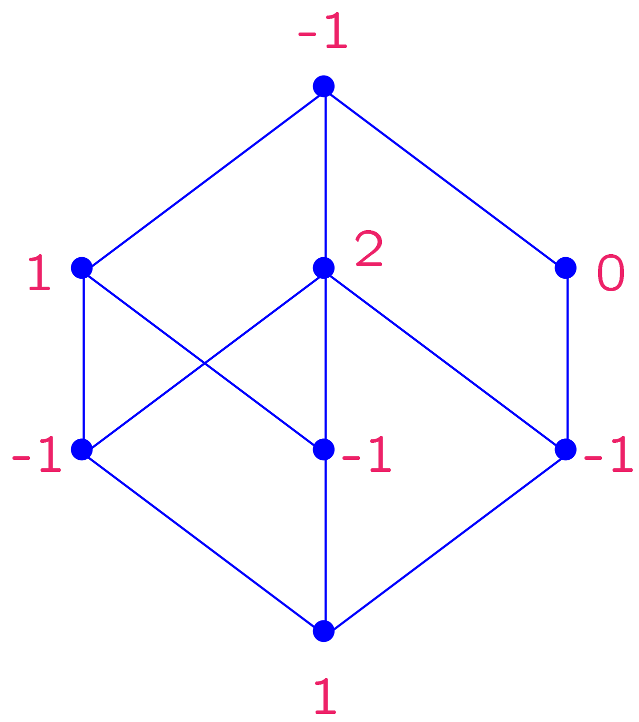

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41756823

3124

41756823

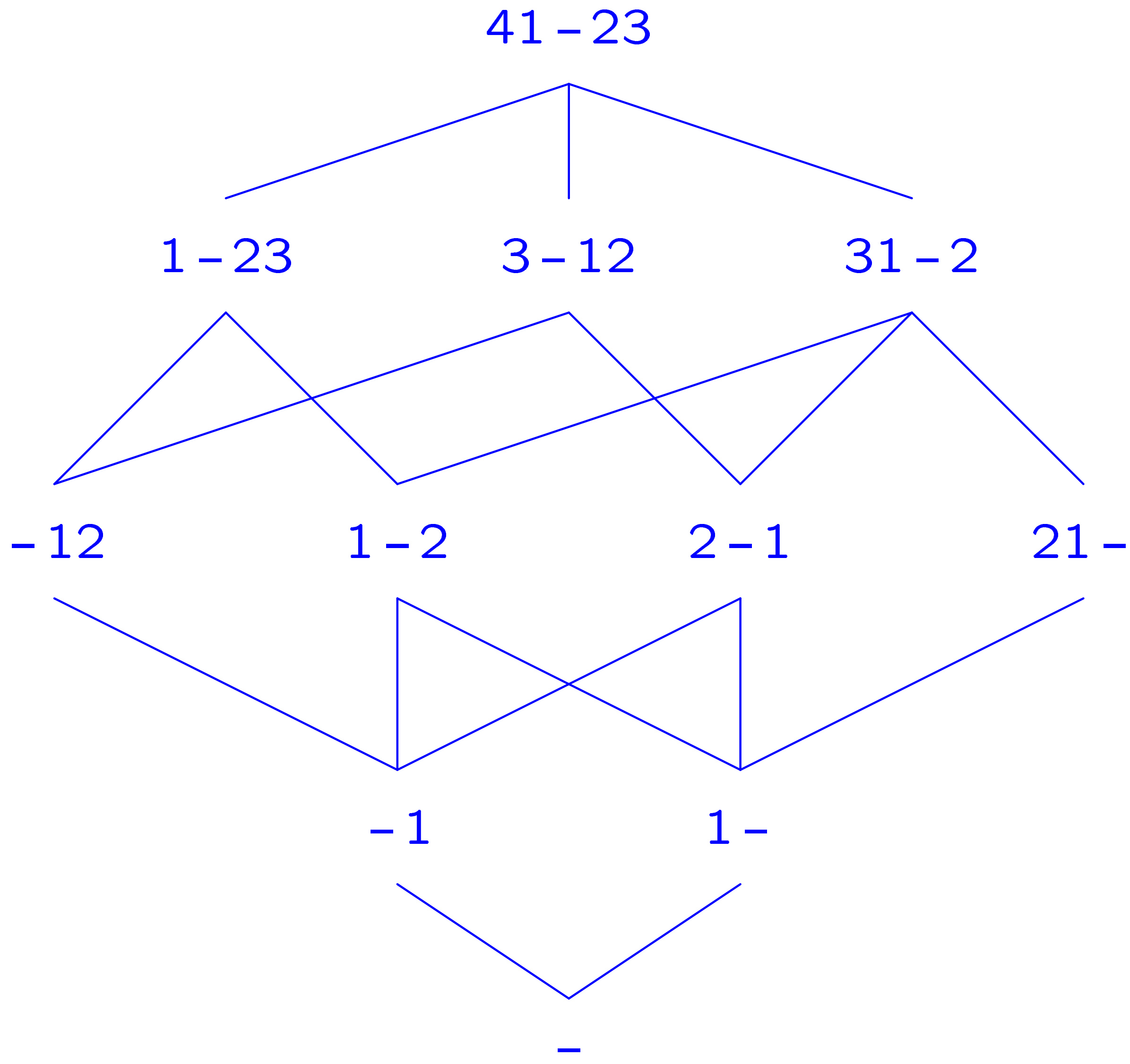
Only one occurrence

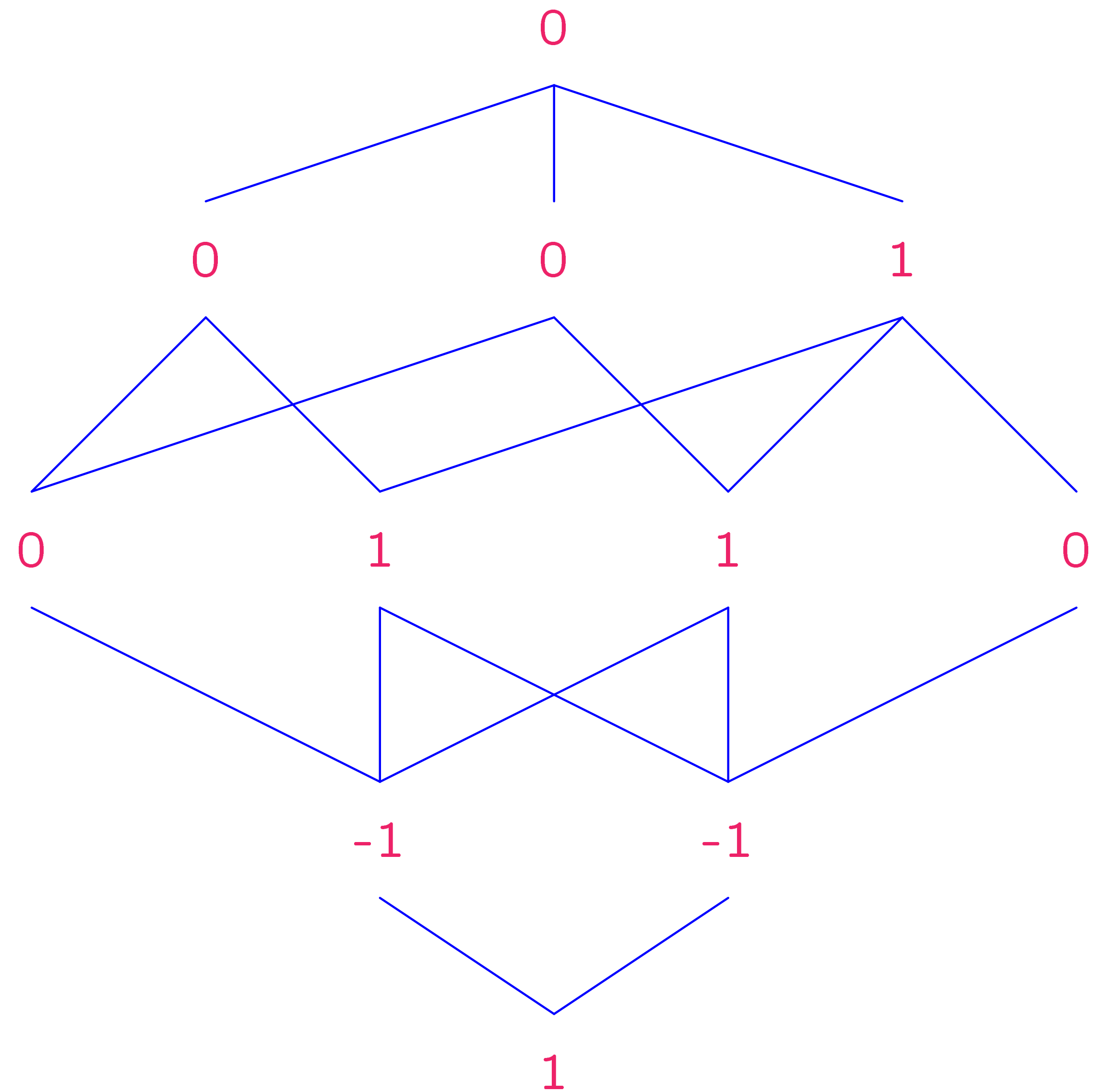
3124

41 – 23

Only one occurrence

–


41-23
1-23
3-12
31-2
-12
1-2
2-1
21-
-1
1-
-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1
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1
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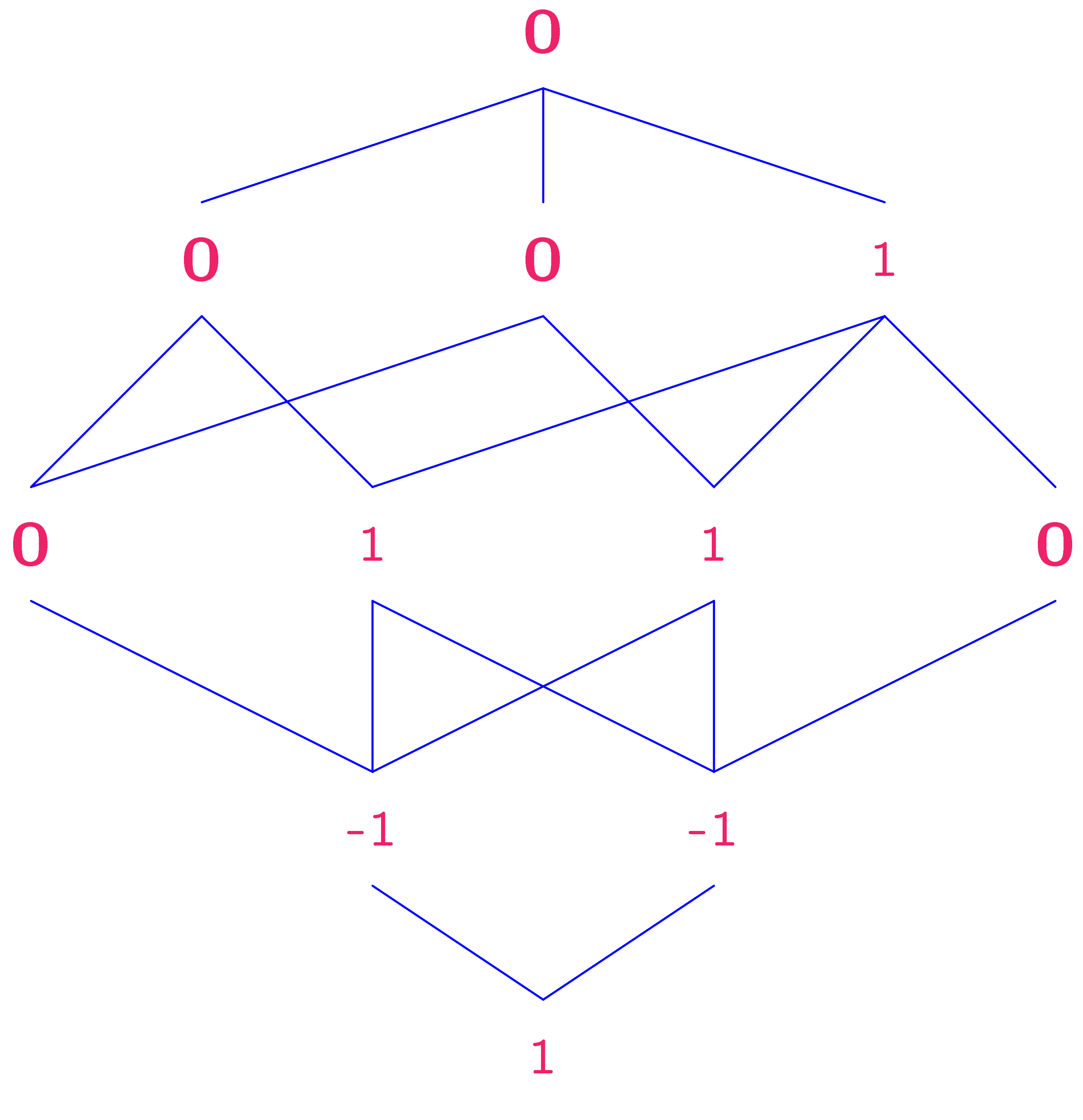
-1
-1
1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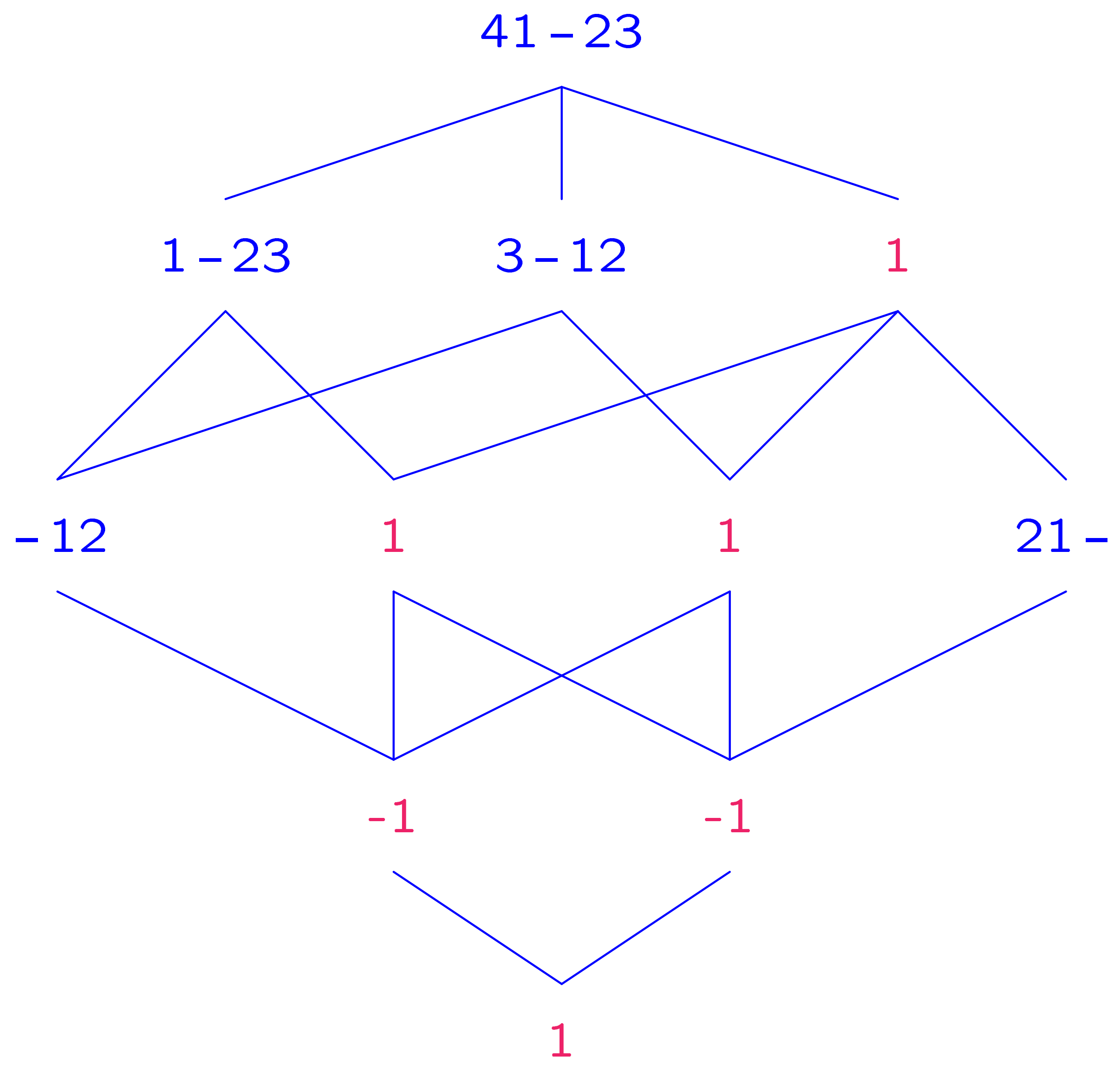
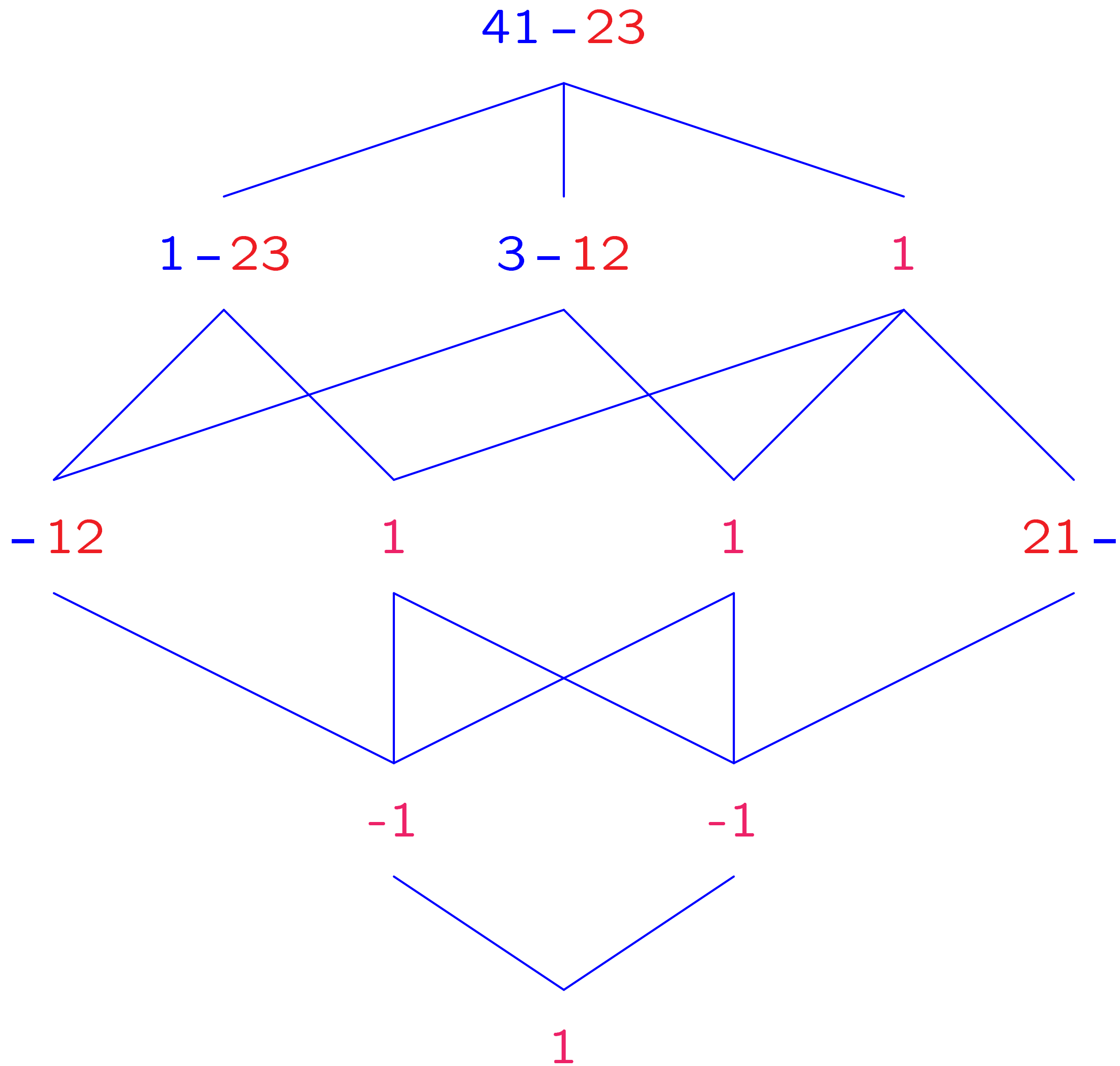
-1
-1
1



41-23
1-23
3-12
1
-12
1
1
21-
-1
-1
1

Interval blocks



The *complement of* σ *in* τ consists of the blocks of contiguous letters in τ that do not belong to any occurrence of σ.

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The occurrences of 123 in 879416235:

879416235

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An *interval block* is a segment of length at least 2 containing the letters $\{k, k+1, \ldots, k+m\}$ for some k and m.

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8794 – 6 –

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A *simple* permutation π is a permutation whose only interval block is π itself.

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The occurrences of 123 in 879416235:

8794 – 6 –

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Theorem: If the complement of σ in τ contains an interval block, then $\mu(\sigma, \tau) = 0$.

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$$\mu(123, 879416235) = 0$$

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The Möbius function of the product of two posets is the product of their MF's, and the Möbius function of a chain with at least 3 elements is 0.

316254

321

316254

Only one occurrence

321

316254

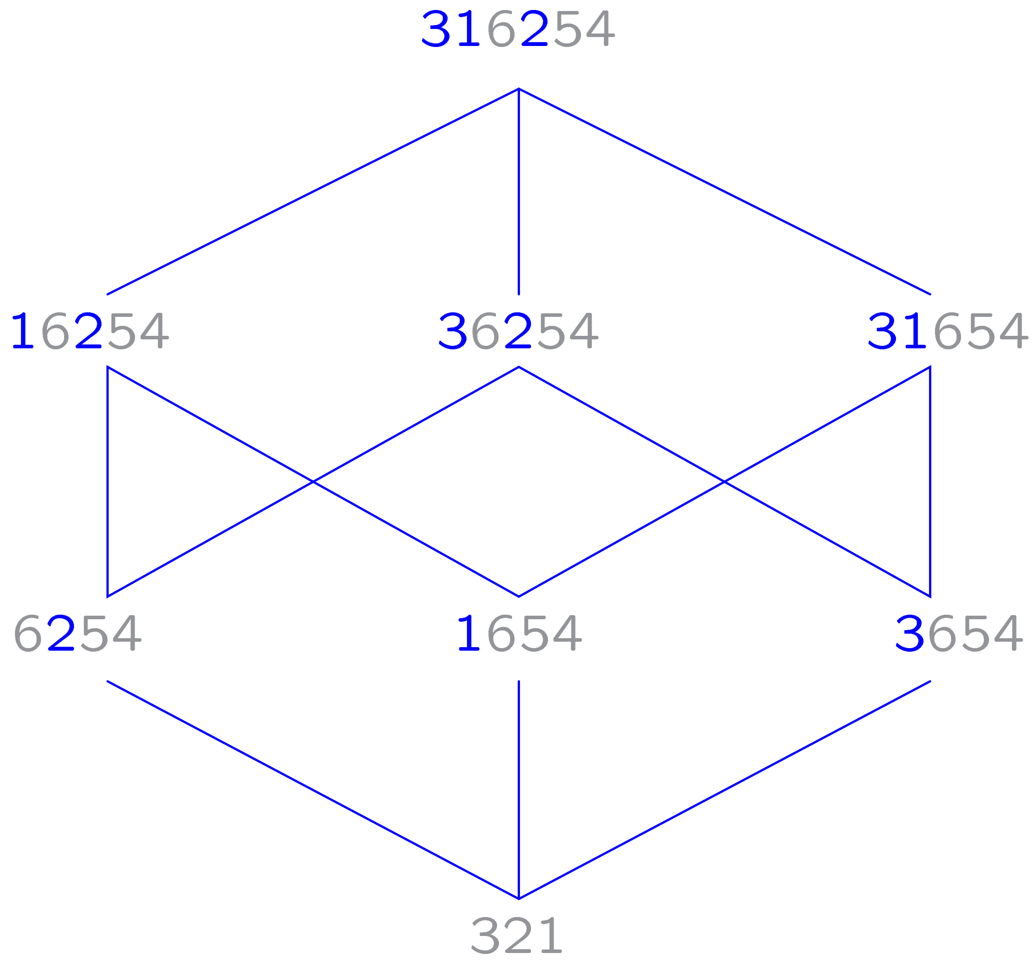
Only one occurrence

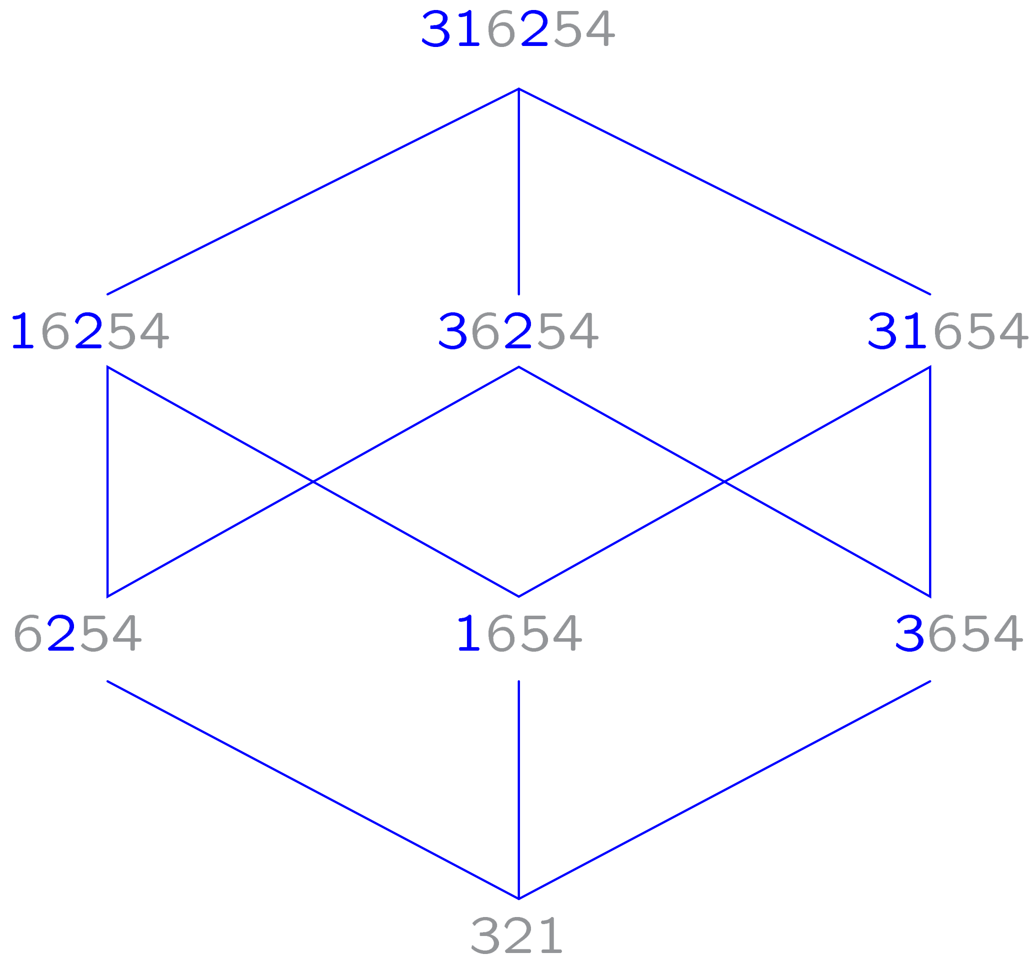
321

316254

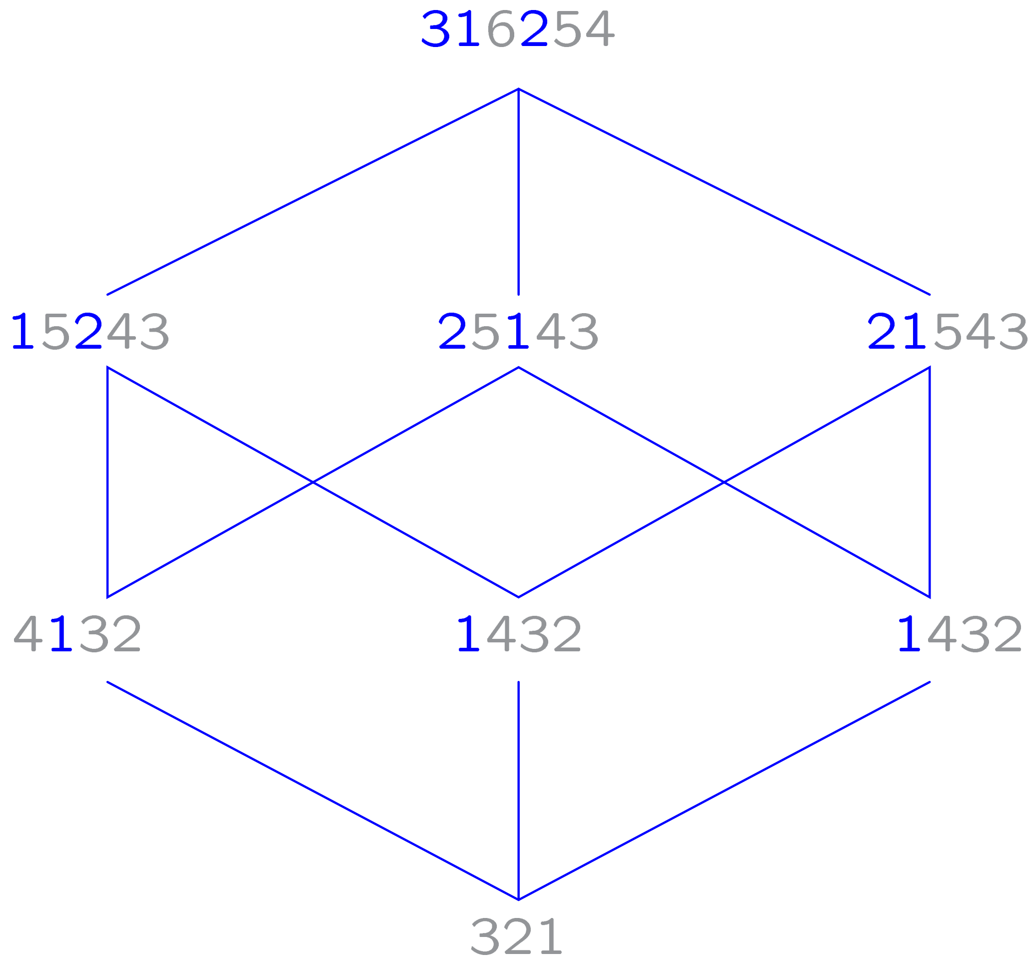
Only one occurrence — construct boolean algebra on complement

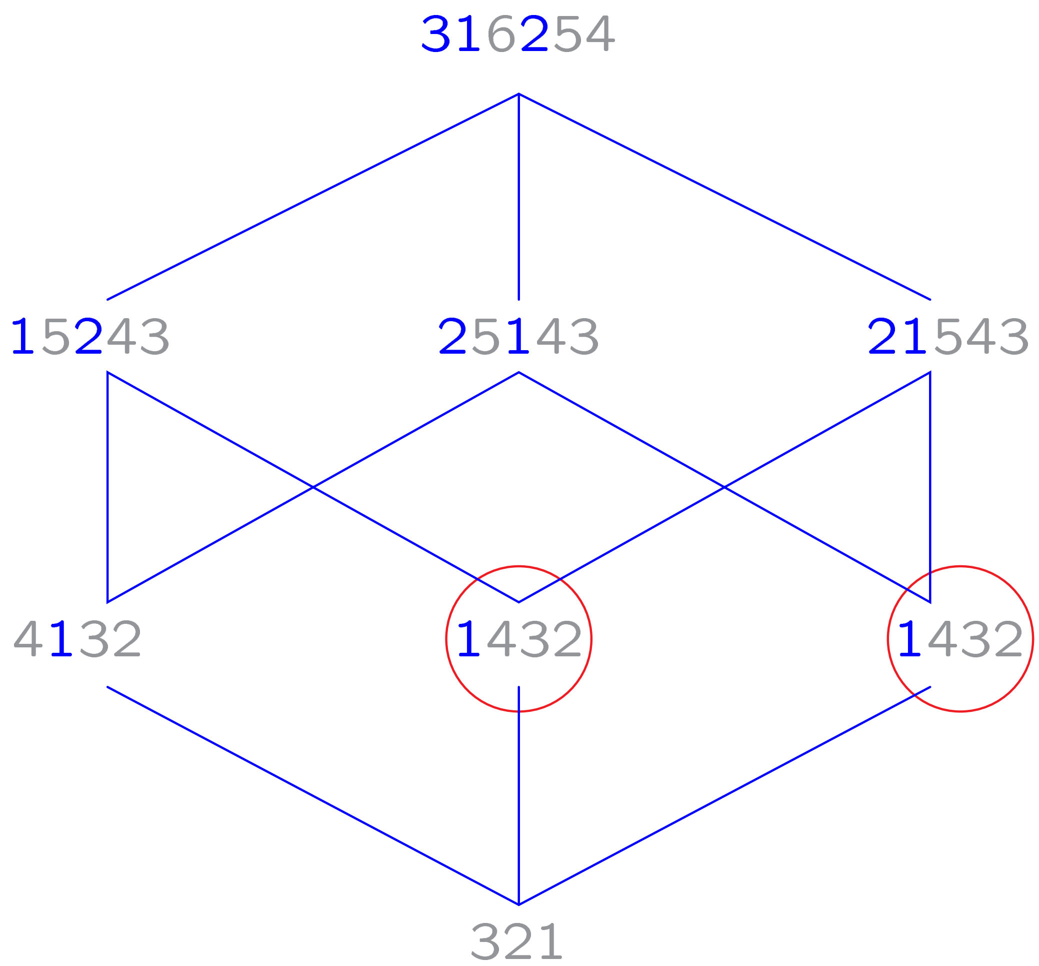
321


316254
16254
36254
31654
6254
1654
3654
321



To get actual elements of the interval, reduce to standard form


316254
15243
25143
21543
4132
1432
1432
321


316254
15243
25143
21543
4132
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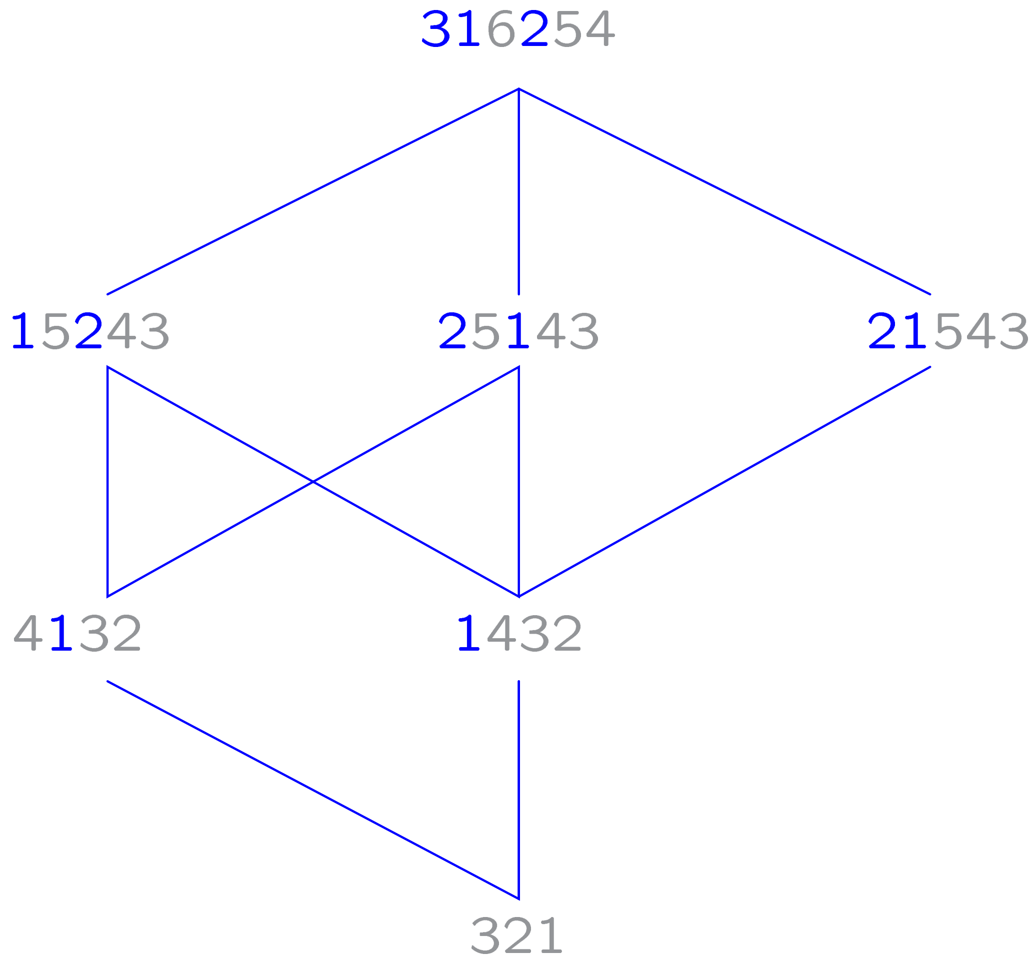
321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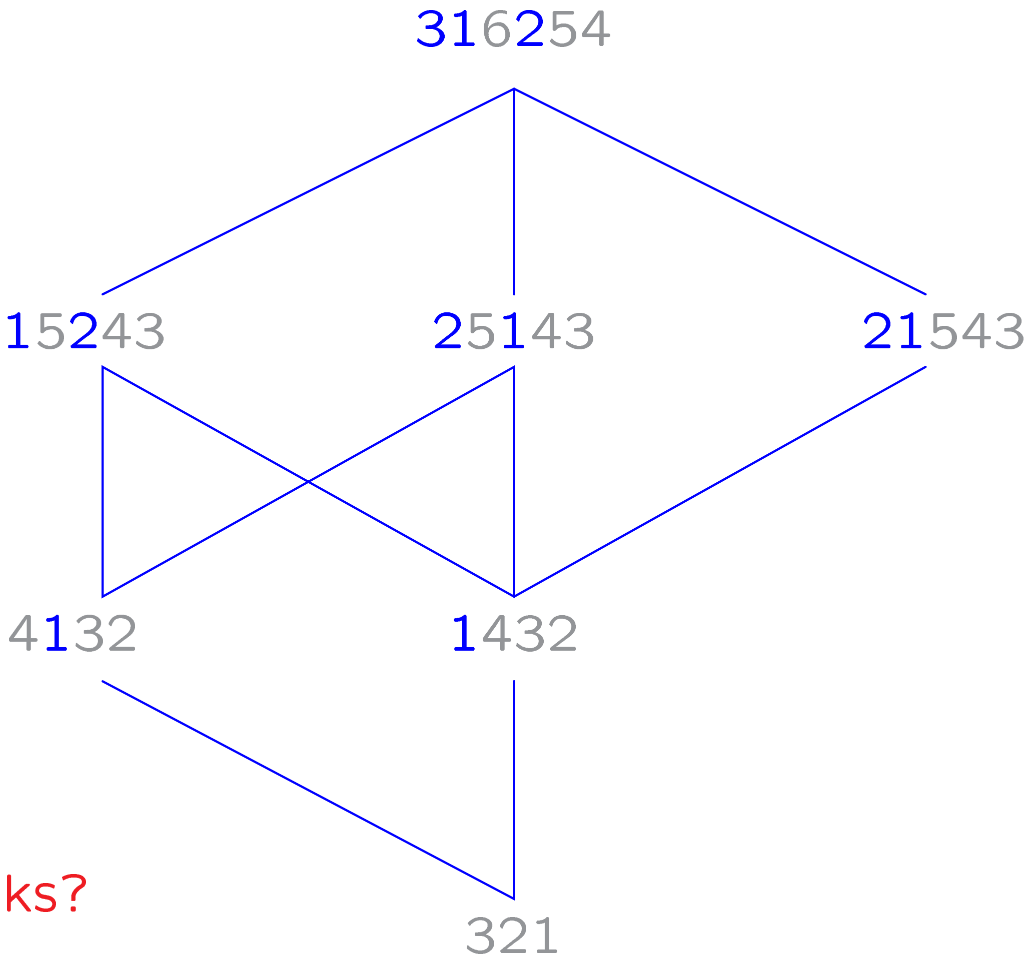
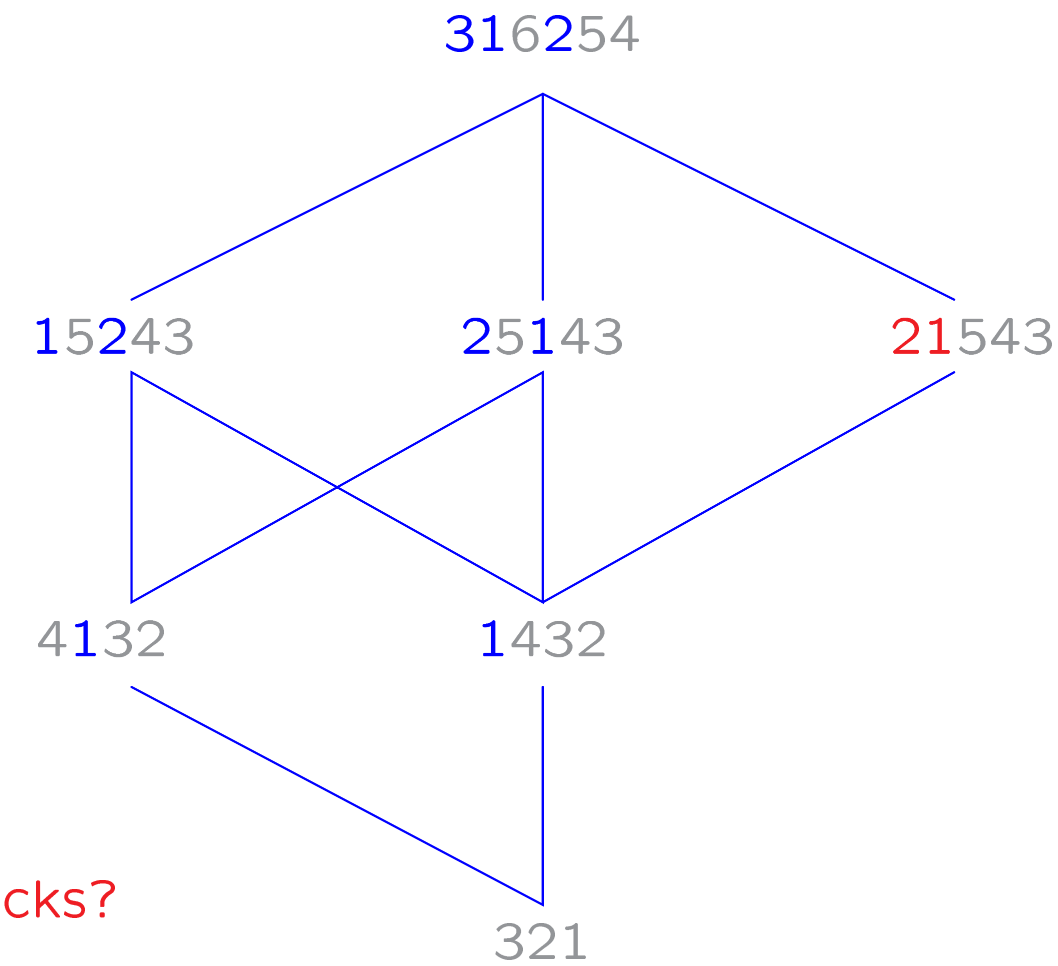


Interval blocks?



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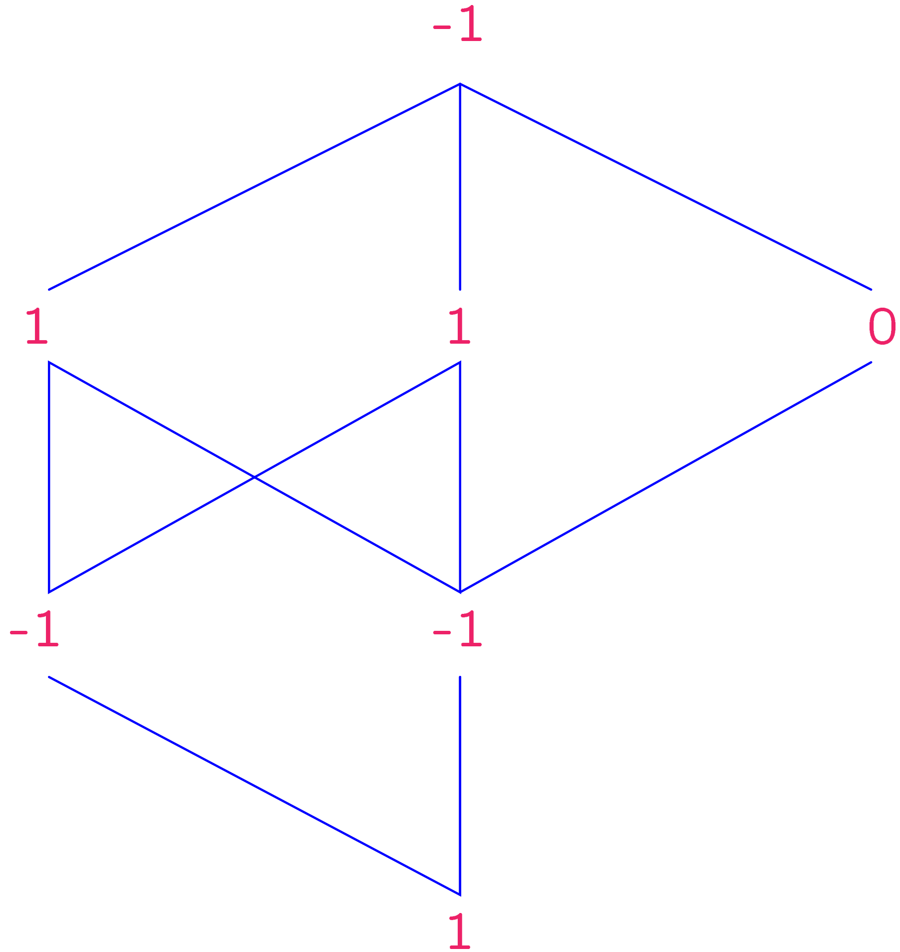
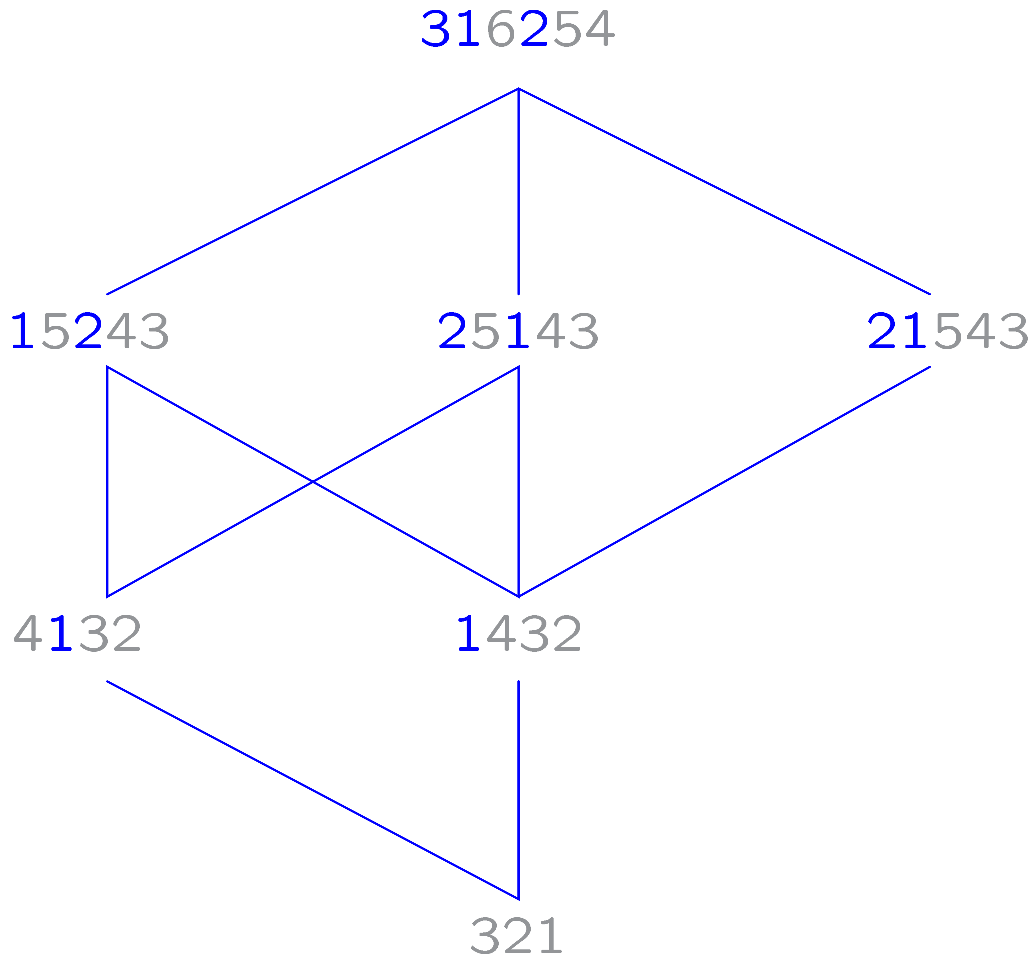

-1
1
1
0
-1
-1
1


316254
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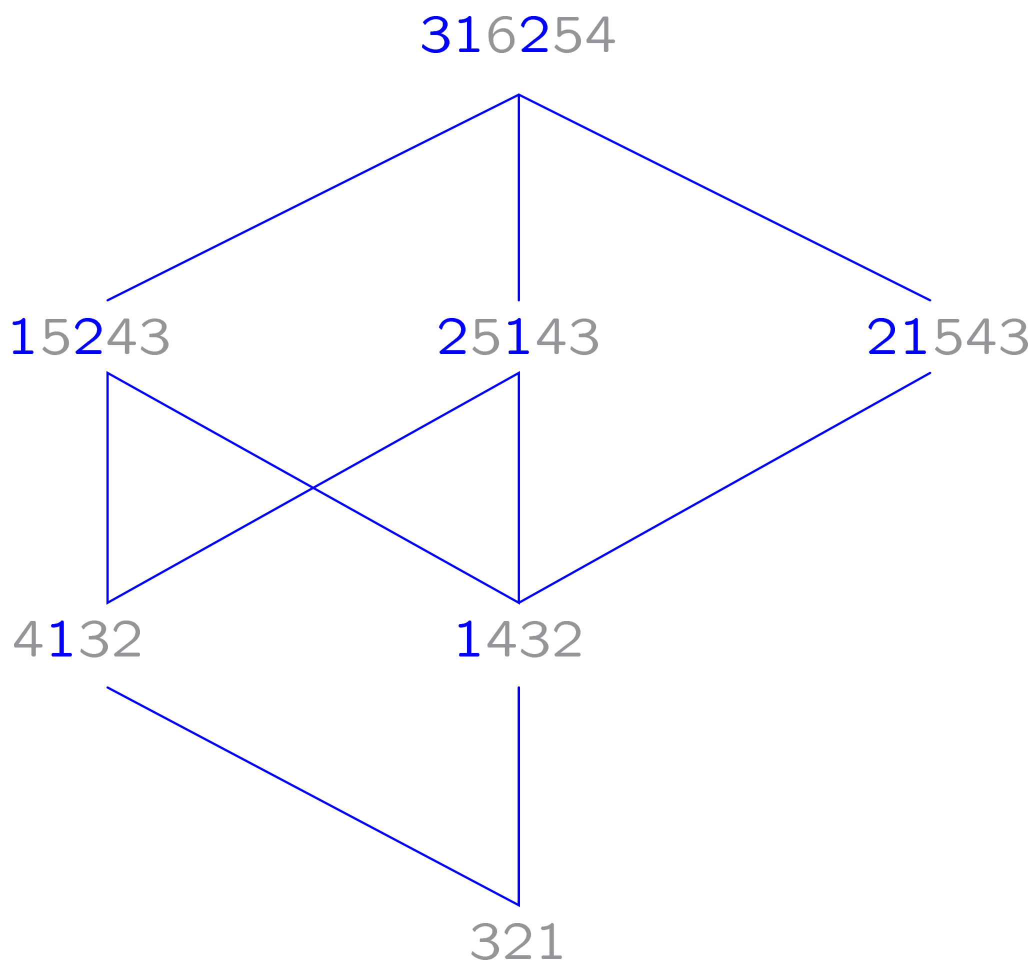
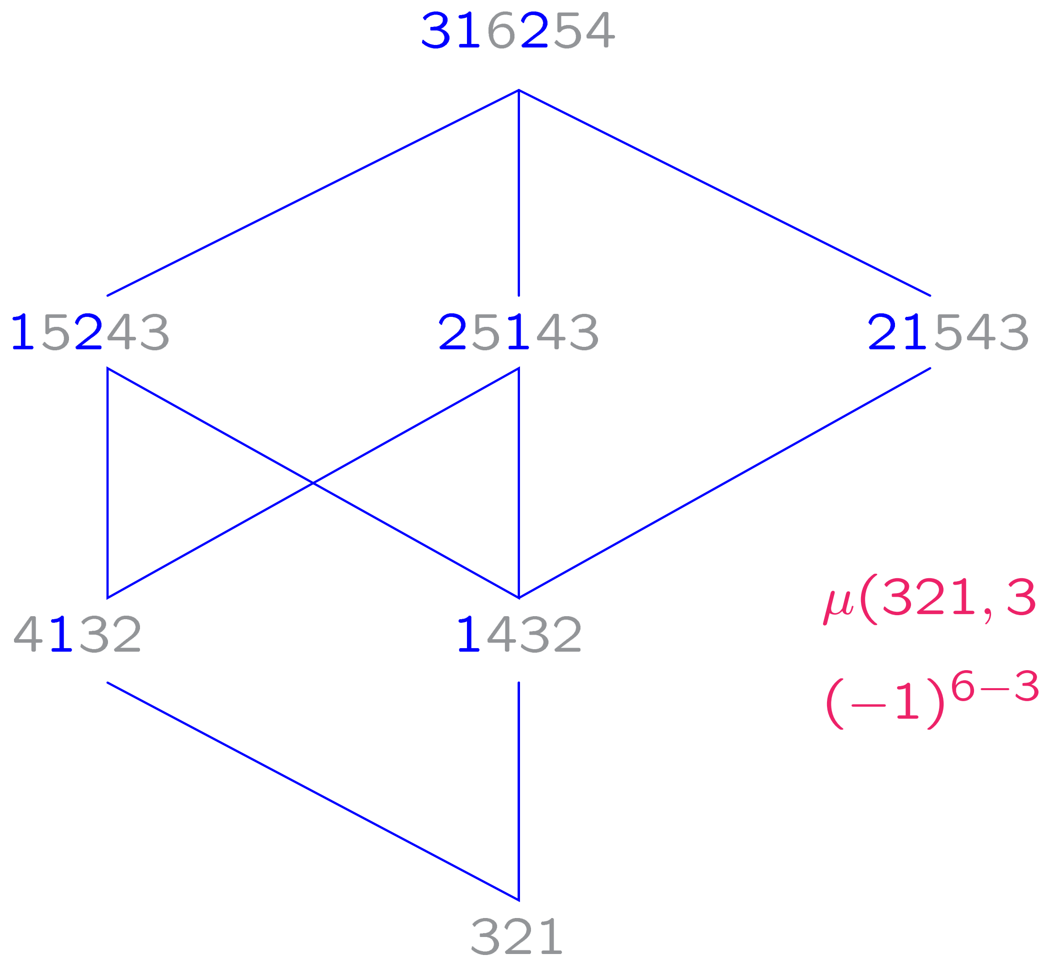


Theorem: Suppose σ occurs precisely once in τ, and that the complement of σ in τ has no interval blocks. Then $\mu(\sigma, \tau) = (-1)^{|\tau|-|\sigma|}$.



$\mu(321, 316254) =$
$(-1)^{6-3} = -1$

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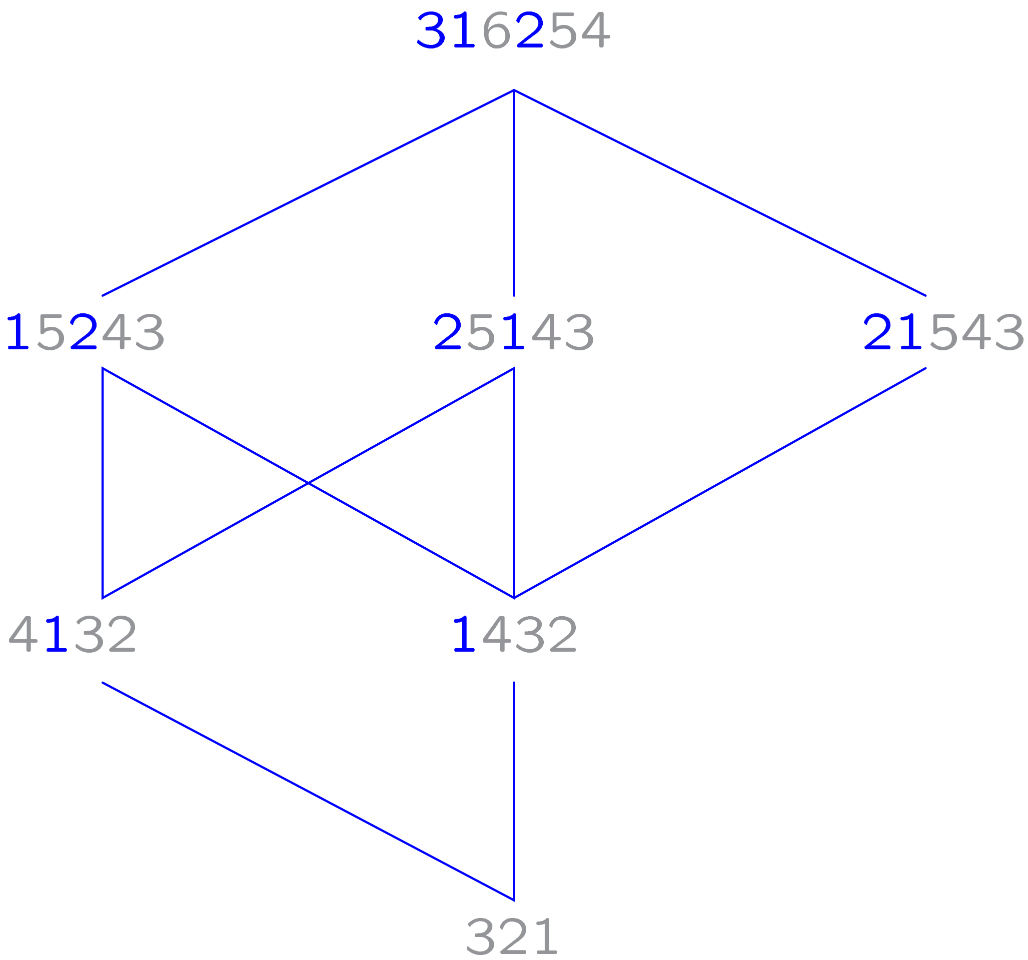
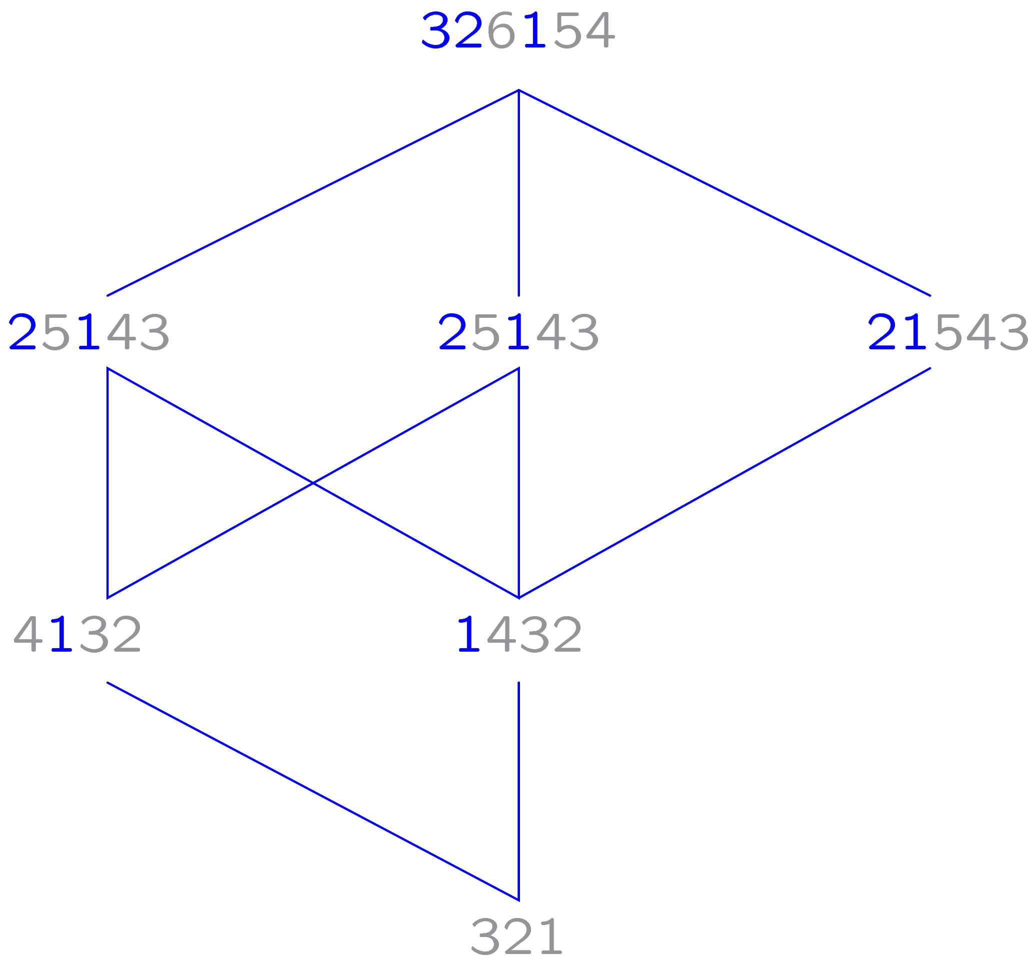
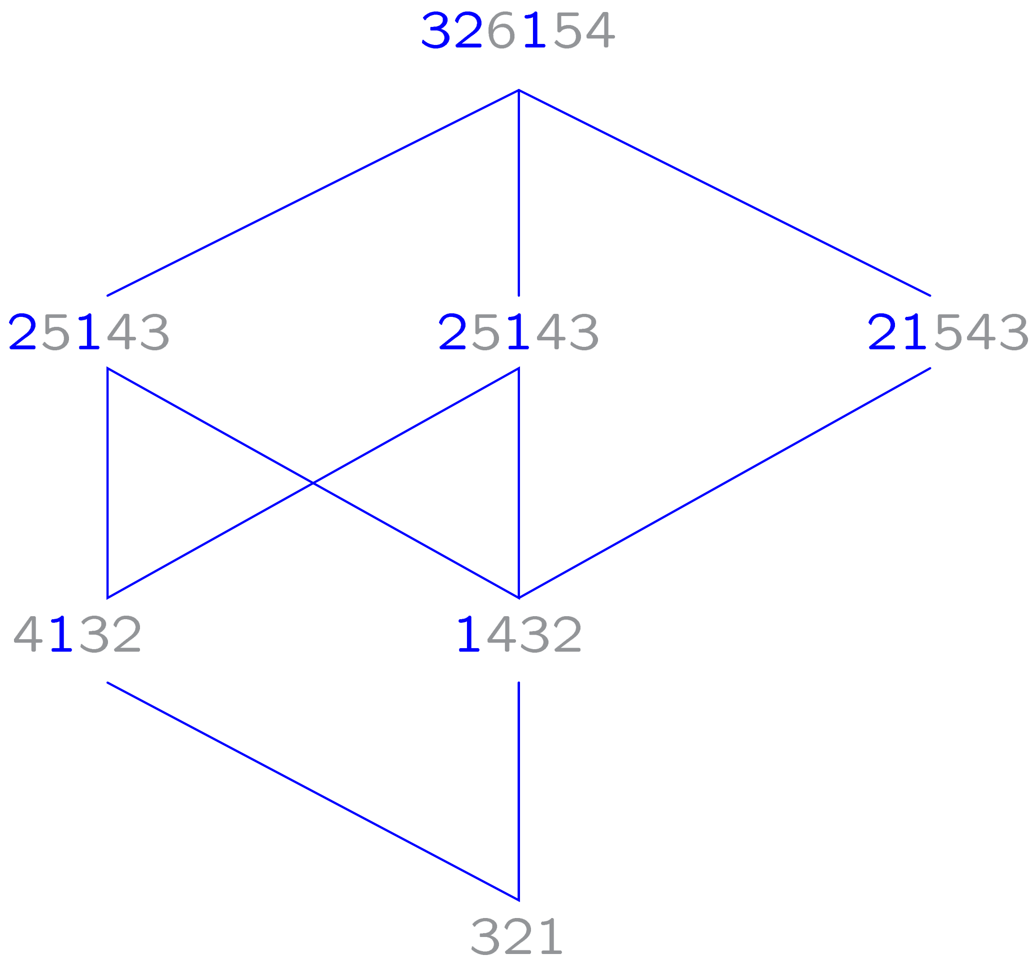


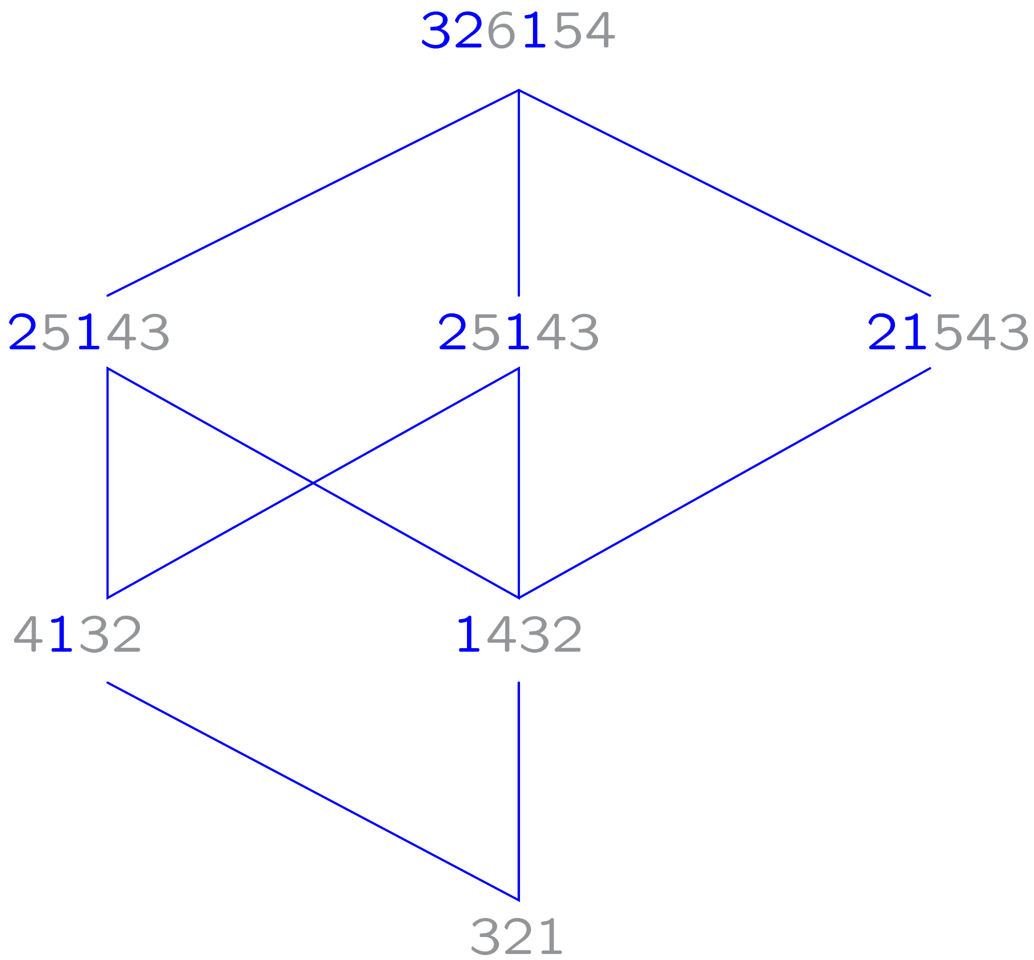
Same poset. . .



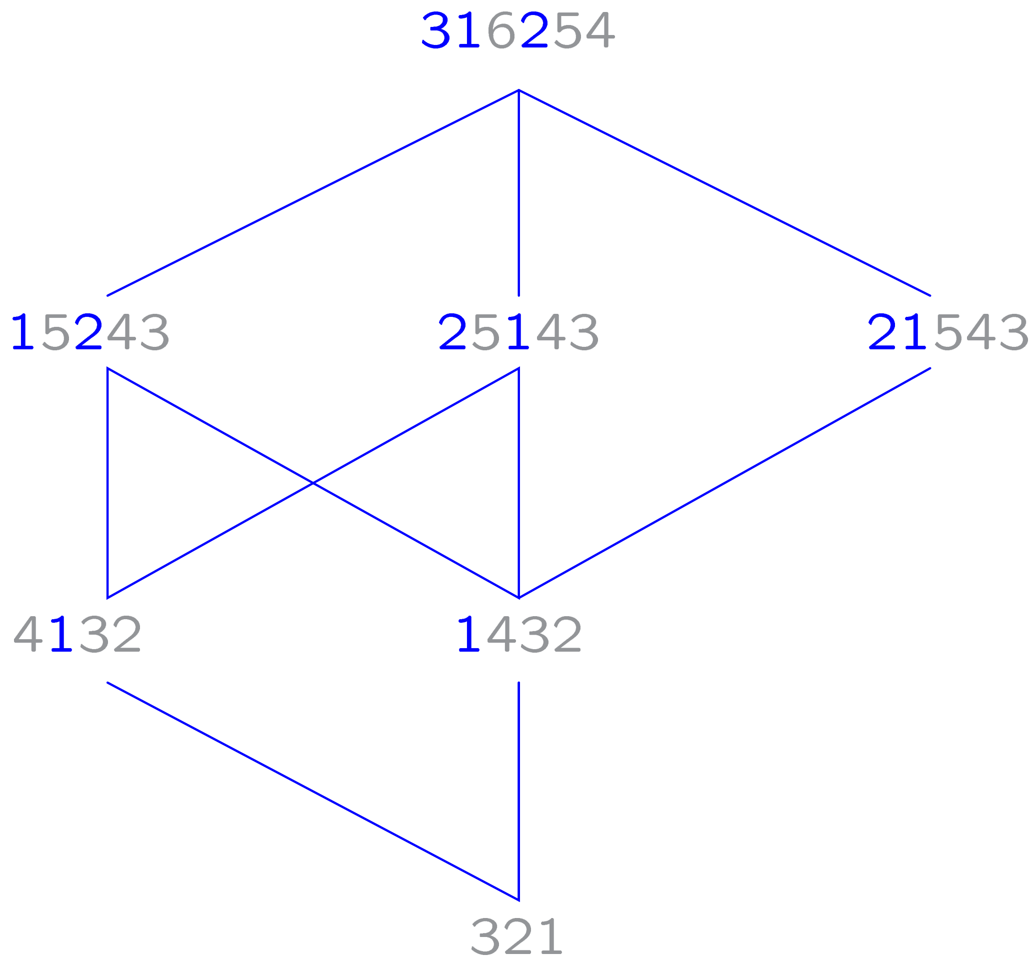
Now there are two occurrences.



Now there are two occurrences. But, . . .

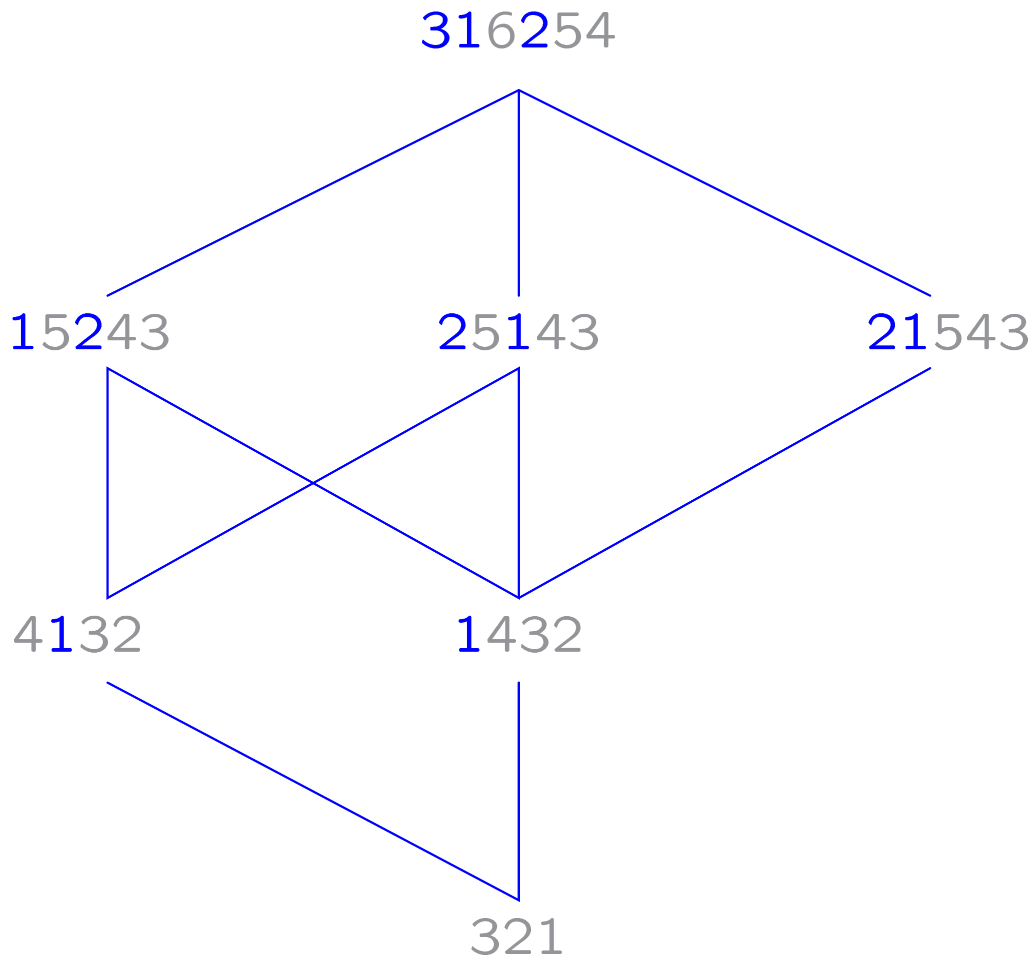


Now there are two occurrences. But, we can still freeze the letters 4,5,6 as before, construct the Boolean algebra on the rest, and obtain an isomorphic poset.



Theorem and Def: Let $<\sigma>$ be a particular occurrence of σ in τ, and let $[<\sigma>, t]$ be the *occurrence poset* defined as above. Suppose there is no interval block in the complement of $<\sigma>$ in τ. Then $\mu([<\sigma>, \tau]) = (-1)^{|\tau|-|\sigma|}$.

$\widehat{P}$:



Theorem (Philip Hall): The Möbius function of $\widehat{P}$ is given by $\sum_i (-1)^i C_i$, where C_i is the number of chains of length i in $\widehat{P}$ that contain both the minimal and maximal elements.

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Lemma: Suppose there are two occurrences of σ in τ. Let S_1 and S_2 be the two occurrence posets. Then

$$\mu(\sigma, \tau) = \mu(S_1) + \mu(S_2) - \mu(S_1 \cap S_2)$$

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Proof: Count chains.

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Generalizes to any number of occurrences.

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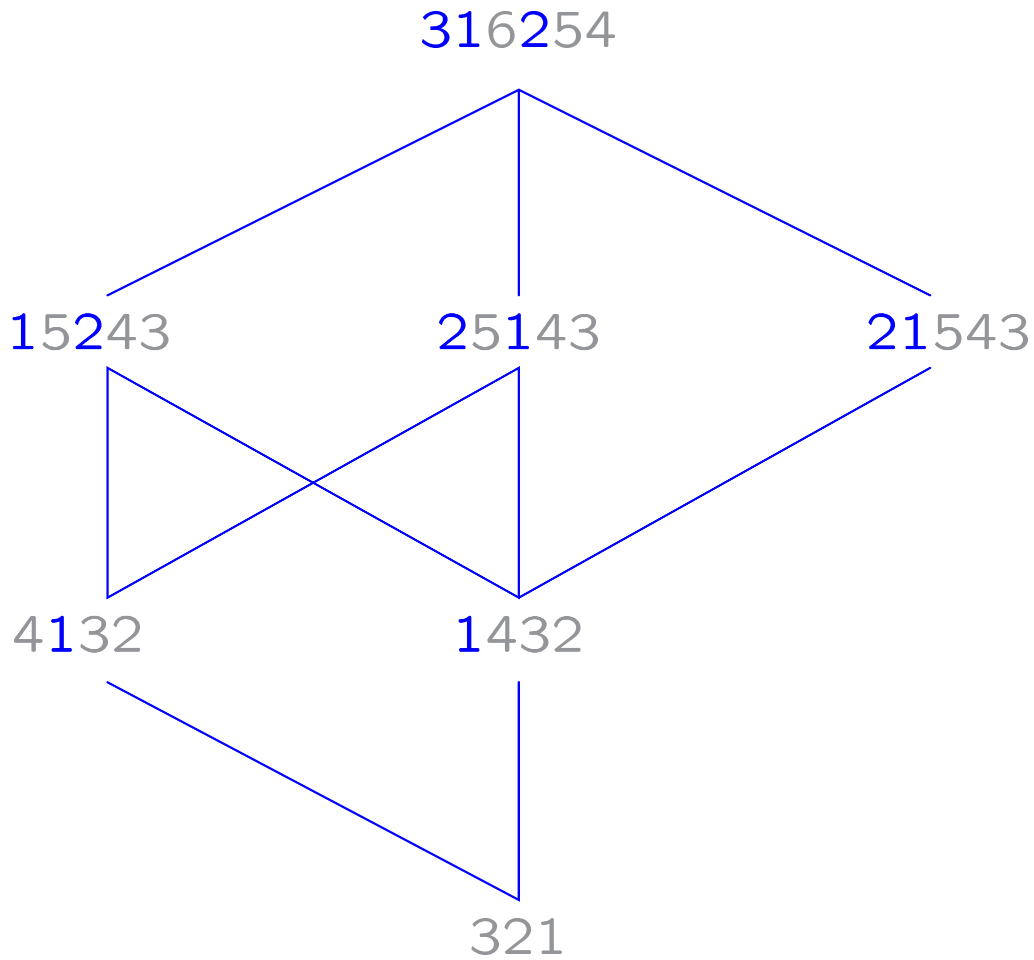
So, if the intersections are nice...

$\widehat{P}$:



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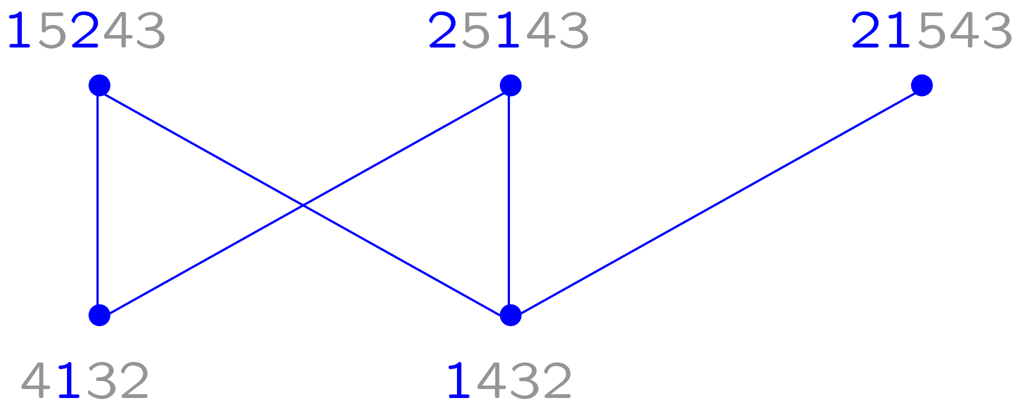
P:



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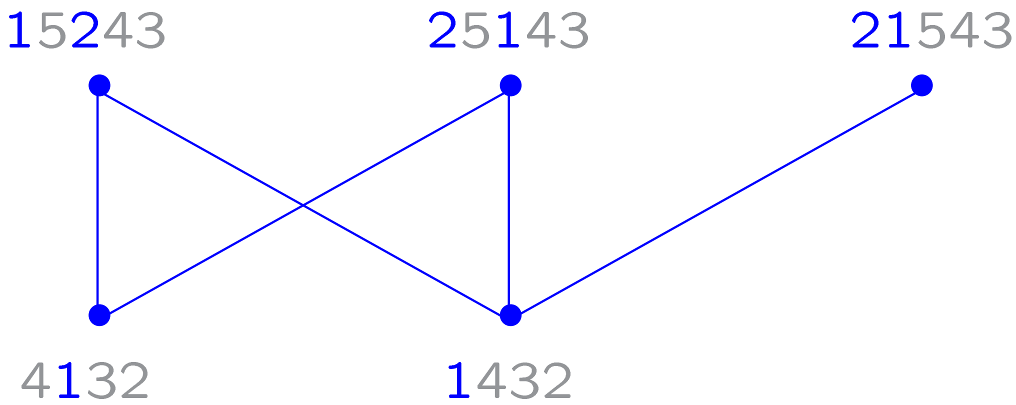
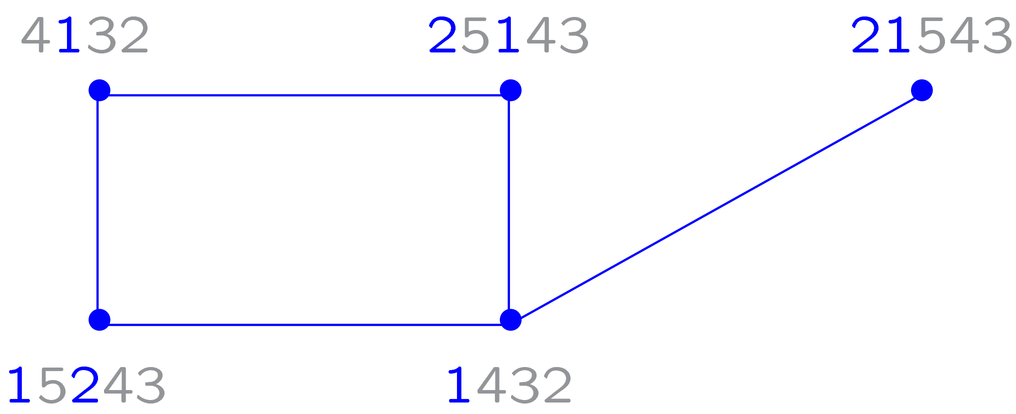
P:



Equivalently: The Möbius function of $\widehat{P}$ equals the (reduced) *Euler characteristic* of the *order complex* of P.

$\Delta(P)$:



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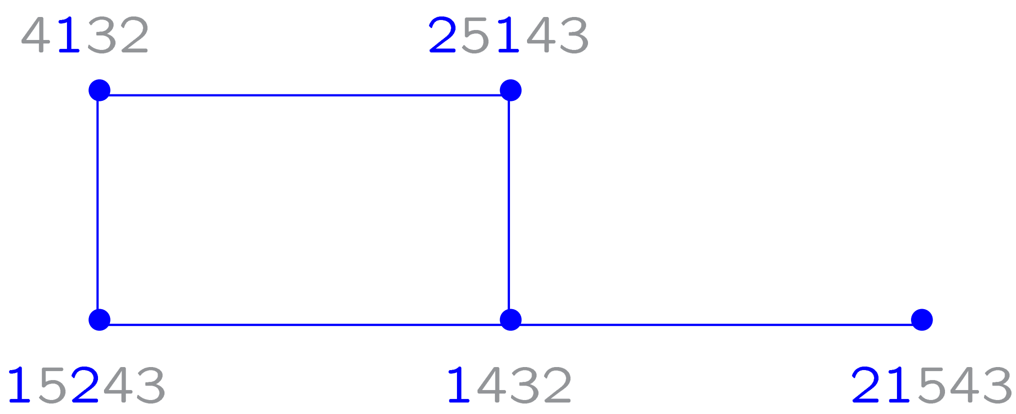

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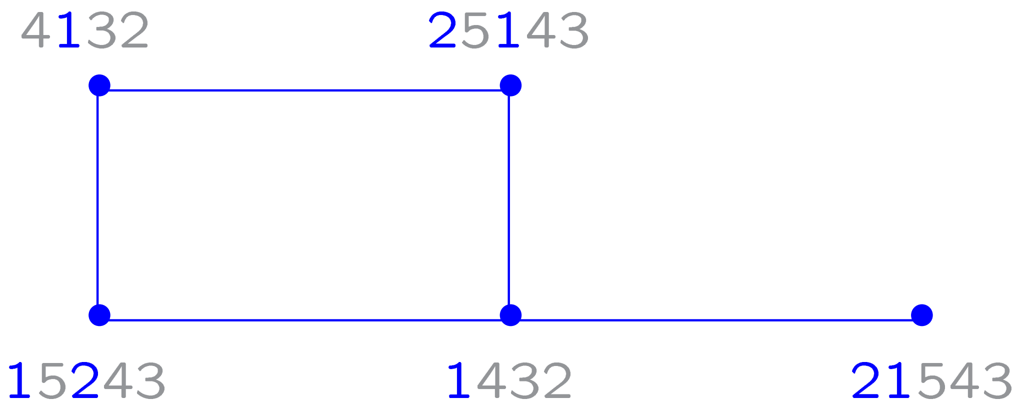
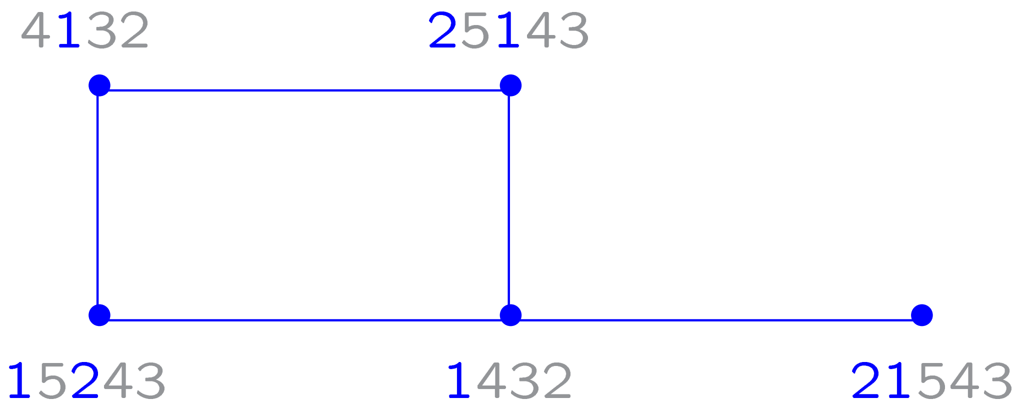


Fact: The Möbius function of $\widehat{P}$ equals $\widetilde{\chi}(\Delta(P))$.

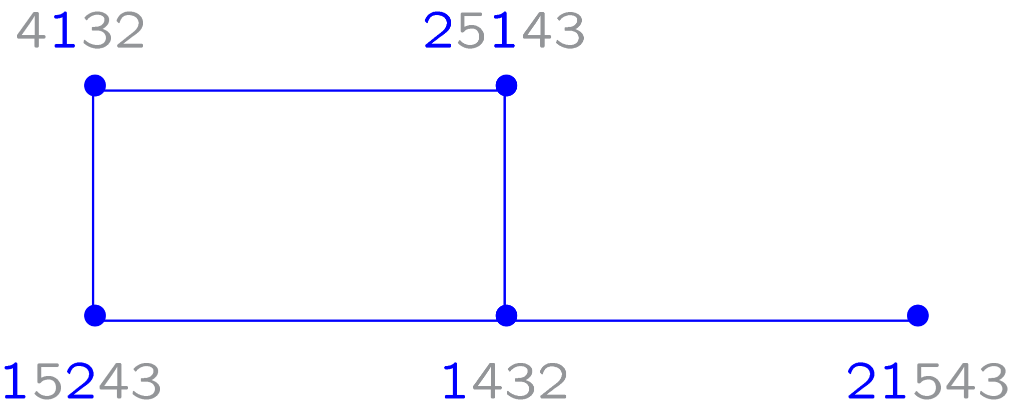
Thus: The Möbius function of $\widehat{P}$ depends only on the homotopy type/homology of $\Delta(P)$.

$\Delta(P)$:



If we can compute the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$, then $h_d = \pm\mu(\sigma, \tau)$.

$\Delta(P)$:



If we can compute the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$, then $h_d = \pm\mu(\sigma, \tau)$.

If $\Delta(P)$ is *shellable* then a shelling may give the h-vector.

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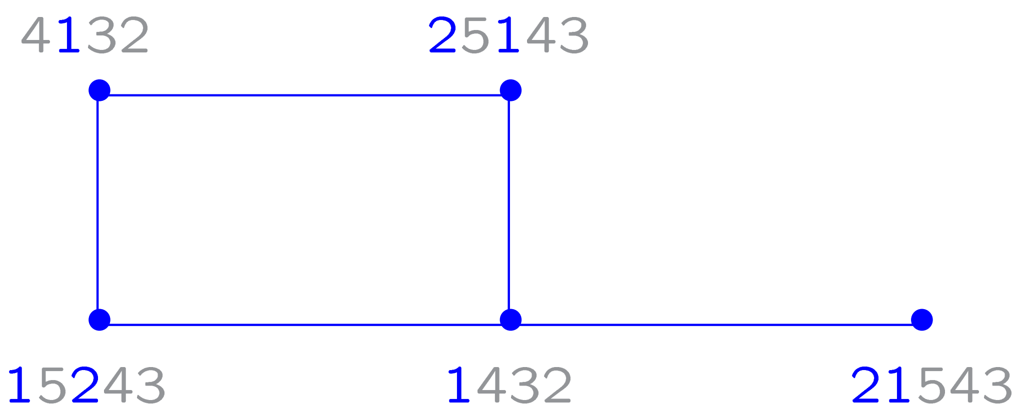

If we can compute the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$, then $h_d = \pm\mu(\sigma, \tau)$.

If $\Delta(P)$ is *shellable* then a shelling may give the h-vector.

$\Delta(P)$ is not always shellable. . .

Conjectures:

Eric Babson:

Eric Babson: “I always prefer being scooped to feeling I ought to be writing things myself.”

Let $\widehat{P} = [\sigma, \tau]$ and suppose σ occurs precisely once in τ

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Conjectures:

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Conjectures:

- The subcomplex of $\Delta(P)$ induced by elements of P with nonzero Möbius function is a sphere

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Conjectures:

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- $\Delta(P)$ has the homotopy type of a sphere

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Conjectures:

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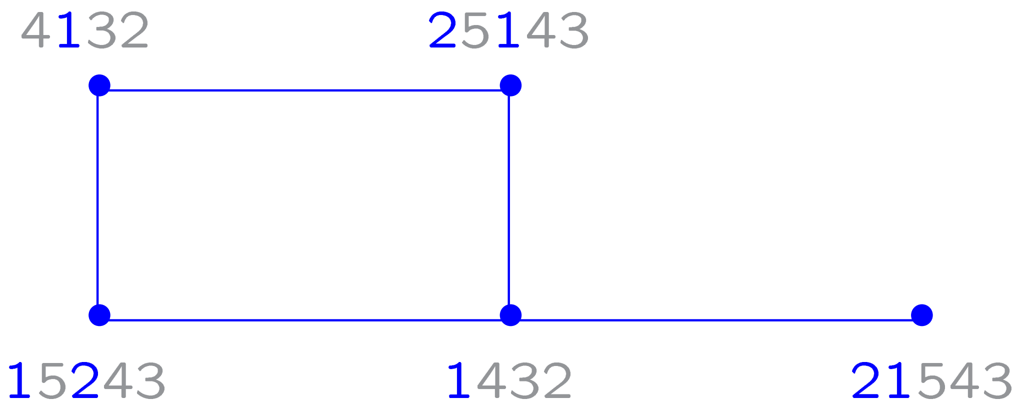
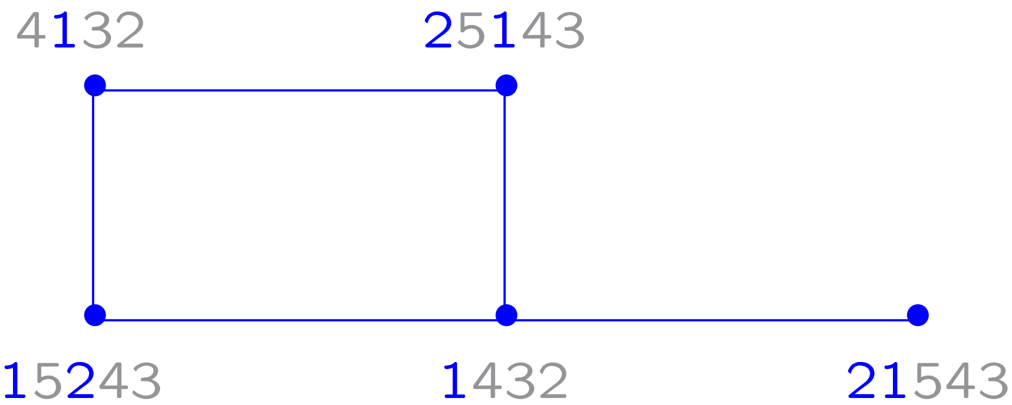
Either conjecture implies theorem:

$$\mu([<\sigma>, \tau]) = (-1)^{|\tau|-|\sigma|}$$


4132
25143
15243
1432
21543



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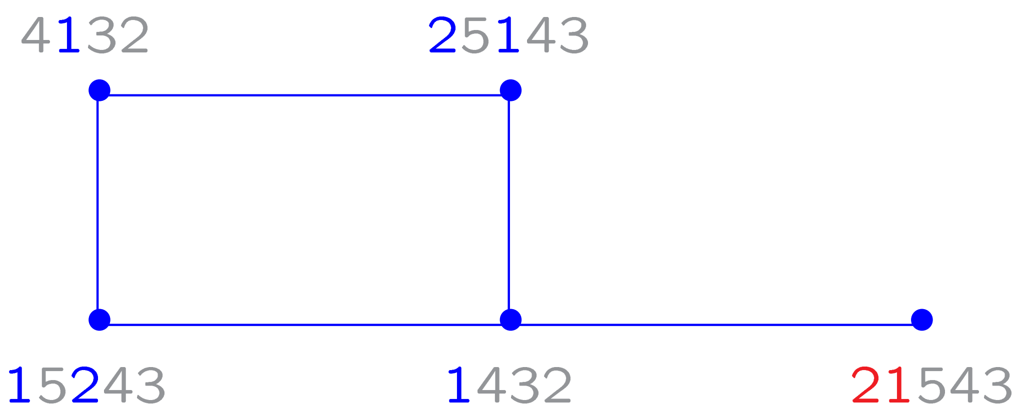

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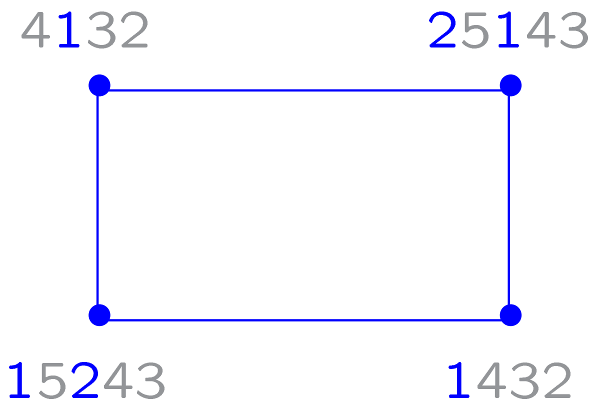

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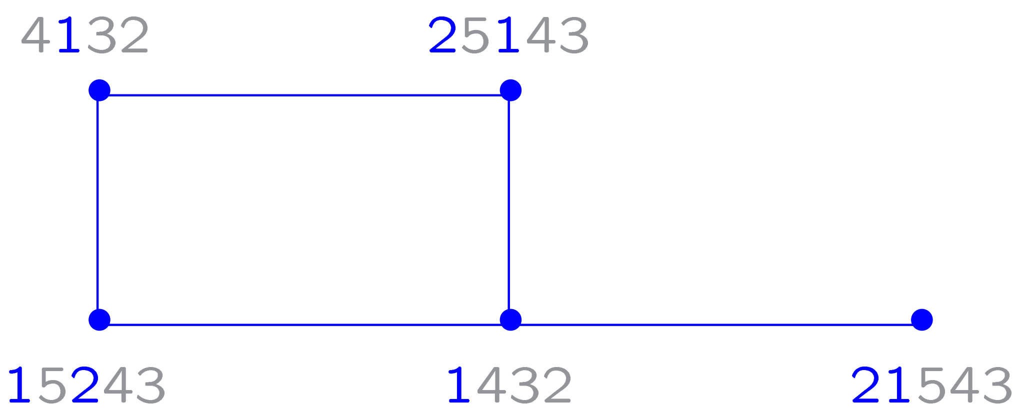

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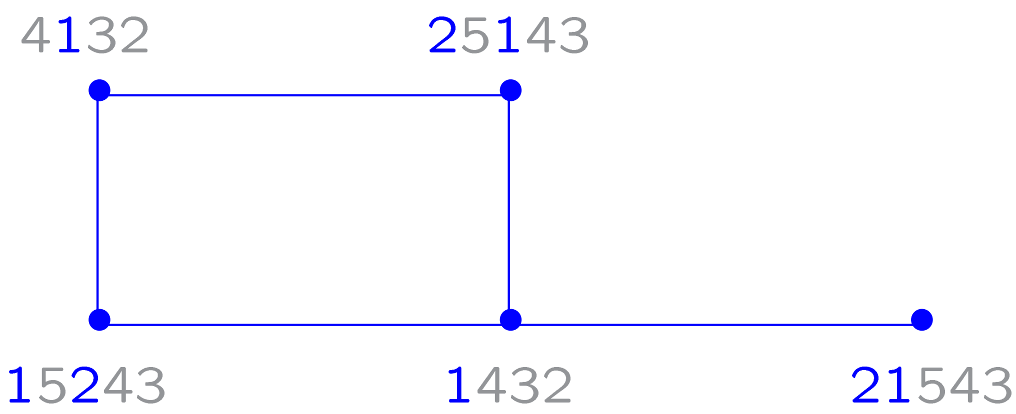

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Let $\widehat{P} = [\sigma, \tau]$ and suppose σ occurs precisely once in τ

Conjectures:

- The subcomplex of $\Delta(P)$ induced by elements of P with nonzero Möbius function is a sphere
- $\Delta(P)$ has the homotopy type of a sphere

Let $\widehat{P} = [\sigma, \tau]$ and suppose σ occurs precisely once in τ

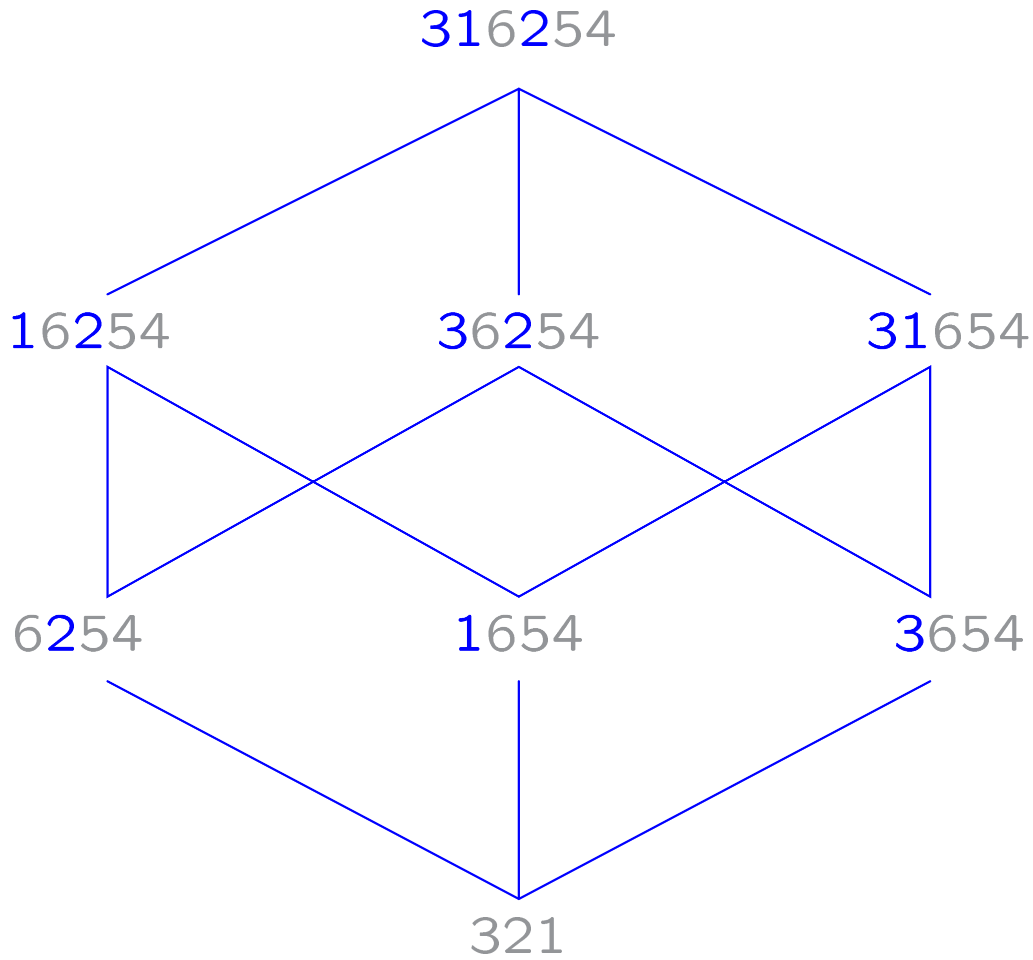
Conjectures:

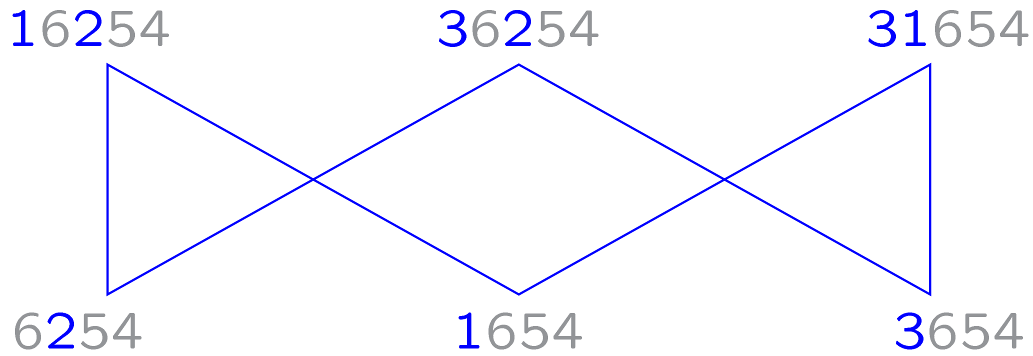
- The subcomplex of $\Delta(P)$ induced by elements of P with nonzero Möbius function is a sphere
- $\Delta(P)$ has the homotopy type of a sphere
- $\Delta(P)$ is shellable

Let $\widehat{P} = [\sigma, \tau]$ and suppose σ occurs precisely once in τ

Conjectures:

- The subcomplex of $\Delta(P)$ induced by elements of P with nonzero Möbius function is a sphere
- $\Delta(P)$ has the homotopy type of a sphere
- $\Delta(P)$ is shellable (at least the nonzero part)


316254
16254
36254
31654
6254
1654
3654
321


16254
36254
31654
6254
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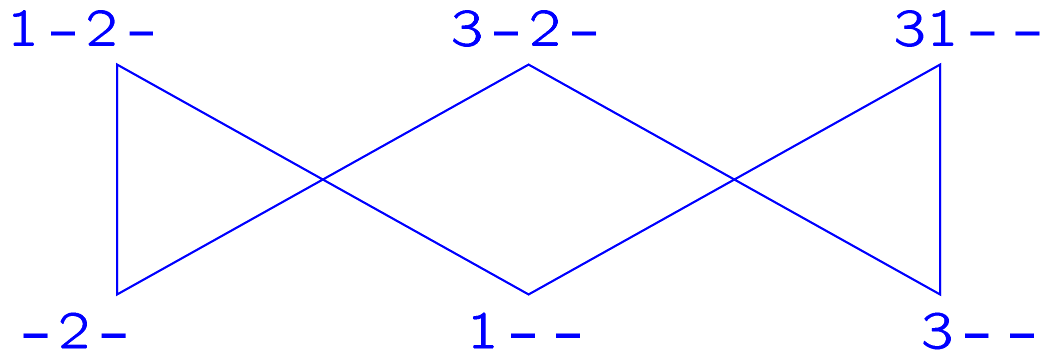
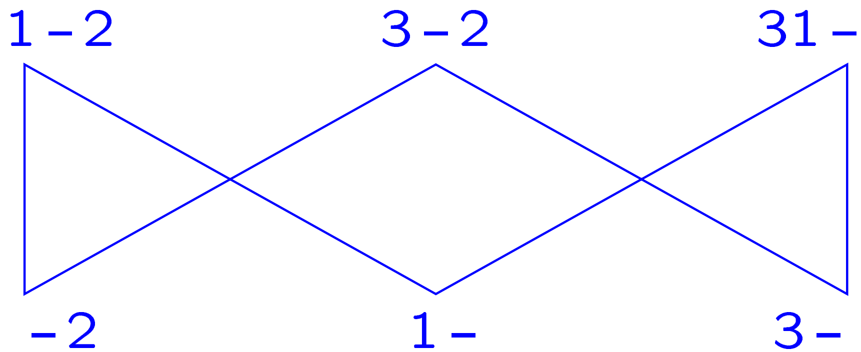
3654


1-2-
3-2-
31--
-2-
1--
3--


1-2
3-2
31-
-2
1-
3-

Why care about the Möbius function?

Why care about the Möbius function?

Because it's there. . .

Let $\sigma(\tau)$ be the number of occurrences of σ in τ.

Conjectures:

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- If $\sigma(\tau) = 2$ then $|\mu(\sigma, \tau)| \leqslant 3$

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- If $\sigma(\tau) = 2$ then $|\mu(\sigma, \tau)| \leqslant 3$
- If $\sigma(\tau) = k$ then $|\mu(\sigma, \tau)|) \leqslant 2^k - 1$???

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- If τ avoids 132 then $|\mu(\sigma, \tau))| \leqslant \sigma(\tau)$

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- If $\sigma(\tau) = k$ then $|\mu(\sigma, \tau)|) \leqslant f(k)$
- If τ avoids 132 then $|\mu(\sigma, \tau))| \leqslant \sigma(\tau)$

Can be grossly violated if σ contains 132

Let $\sigma(\tau)$ be the number of occurrences of σ in τ.

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- If $\sigma(\tau) = 2$ then $|\mu(\sigma, \tau)| \leqslant 3$
- If $\sigma(\tau) = k$ then $|\mu(\sigma, \tau)|) \leqslant f(k)$
- If τ avoids 132 then $|\mu(\sigma, \tau))| \leqslant \sigma(\tau)$. Equivalently:

Let $\sigma(\tau)$ be the number of occurrences of σ in τ.

Conjectures:

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- If $\sigma(\tau) = k$ then $|\mu(\sigma, \tau)|) \leqslant f(k)$
- If τ avoids 132 then $|\mu(\sigma, \tau))| \leqslant \sigma(\tau)$. Equivalently:
- If $|\mu(\sigma, \tau))| > \sigma(\tau)$, then τ contains all of 132, 213, 231, 312

Other open problems:

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- Determine the Möbius function for the poset of consecutive patterns. Being worked on now. . .

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- Determine the Möbius function for the poset of consecutive patterns. Being worked on now. . .
- Determine the Möbius function for the poset of generalized patterns

Other open problems:

- Determine the Möbius function for the poset of consecutive patterns. Being worked on now...
- Determine the Möbius function for the poset of generalized patterns (which poset?)

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- Determine the Möbius function for the poset of consecutive patterns. Being worked on now. . .
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- If σ and τ are *simple* is $\mu(\sigma, \tau) \neq 0$?

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- Determine the Möbius function for the poset of consecutive patterns. Being worked on now...
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- When is $\mu([1, \pi]) = 0$?

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- When is $\mu([1, \pi]) = 0$?
- When is $\mu([\sigma, \tau]) = 0$?