New pattern matching conditions for wreath products of the cyclic groups with symmetric groups

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Itinerary:

- Definitions
- Consecutive Case
  - Generating Functions
  - Sequences involved and conjectures
  - Non-overlapping GF
- Preview of Distributed Case

$C_k \wr S_n$

$C_k \wr S_n$

We can think of the elements $C_k \wr S_n$ as pairs $\gamma = (\sigma, \epsilon)$ where $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ and $\epsilon = \epsilon_1 \ldots \epsilon_n \in \{0, 1, 2, \ldots, k-1\}^n$.

$|C_k \wr S_n| = k^n n!$

Matching on the underlying permutation

Red(σ)

If $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$.

τ*-match at place* i provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$.

Matching on the labels

$Red(w)$

If $w = 2\ 7\ 2\ 4\ 7$, then $\mathrm{red}(w) = 0\ 2\ 0\ 1\ 2$.

u-match at place i provided $\mathrm{red}(w_i \cdots w_{i+j-1}) = u$.

If $(\tau, u) = (1\ 2, 0\ 0)$ and $(\sigma, w) = (1\ 3\ 2\ 4, 1\ 2\ 2\ 2)$, then there are 2 bi-occurrences of (τ, u) in (σ, w).

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There is only 1 bi-match of (τ, u) in (σ, w).

(τ, u)-mch$((\sigma, w))$:= the number of
(τ, u)-bi-matches in $(\sigma, w) \in C_k \wr S_n$.

(τ, u)-nlap$((\sigma, w))$:= the maximum number of non-overlapping
(τ, u)-bi-matches in (σ, w)

First Goal:

Study the distribution of bi-matches for patterns of length 2 and pairs of patterns of length 2.

$$\begin{aligned}
des((\sigma, w)) &= |\{i : \sigma_i > \sigma_{i+1} \ \& \ w_i \geq w_{i+1}\}| \\
wdes((\sigma, w)) &= |\{i : \sigma_i > \sigma_{i+1} \ \& \ w_i = w_{i+1}\}|, \\
sdes((\sigma, w)) &= |\{i : \sigma_i > \sigma_{i+1} \ \& \ w_i > w_{i+1}\}|,
\end{aligned}$$

$i \in WDes((\sigma, w))$ if and only if there is a $(2\ 1, 0\ 0)$-bi-match starting at position i,

$i \in SDes((\sigma, w))$ if and only if there is a $(2\ 1, 1\ 0)$-bi-match starting at position i, and

$i \in Des((\sigma, w))$ if and only if there is a Υ-bi-match starting at position i where $\Upsilon = \{(2\ 1, 0\ 0), (2\ 1, 1\ 0)\}$.

$$\begin{aligned}
\operatorname{ris}((\sigma, w)) &= \operatorname{des}((\sigma^r, w^r)),\\
\operatorname{wris}((\sigma, w)) &= \operatorname{wdes}((\sigma^r, w^r)), \text{ and}\\
\operatorname{sris}((\sigma, w)) &= \operatorname{sdes}((\sigma^r, w^r)).
\end{aligned}$$

Thm. (KNRR)

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{\mathrm{ris}((\sigma,w))} = \frac{1-x}{1-x+\sum_{n\geq 1}\frac{((x-1)t)^n}{n!}\binom{n+k-1}{n}}.$$

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{\mathrm{wris}((\sigma,w))} = \frac{1-x}{1-x+k(e^{(x-1)t}-1)}.$$

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{\mathrm{sris}((\sigma,w))} = \frac{1-x}{1-x+\sum_{n\geq 1}\frac{((x-1)t)^n}{n!}\binom{k}{n}}.$$

What if $\Upsilon = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$?

In this case we have a Υ-bi-match in (σ, w) starting at i if and only if $\sigma_i < \sigma_{i+1}$ and $w_i \neq w_{i+1}$.

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{(\sigma,w)\in C_k \wr S_n} x^{\Upsilon\text{-mch}((\sigma,w))} = \frac{(k-1)(1-x)}{(k-1)(1-x)+k(e^{(k-1)(x-1)t}-1)}.$$

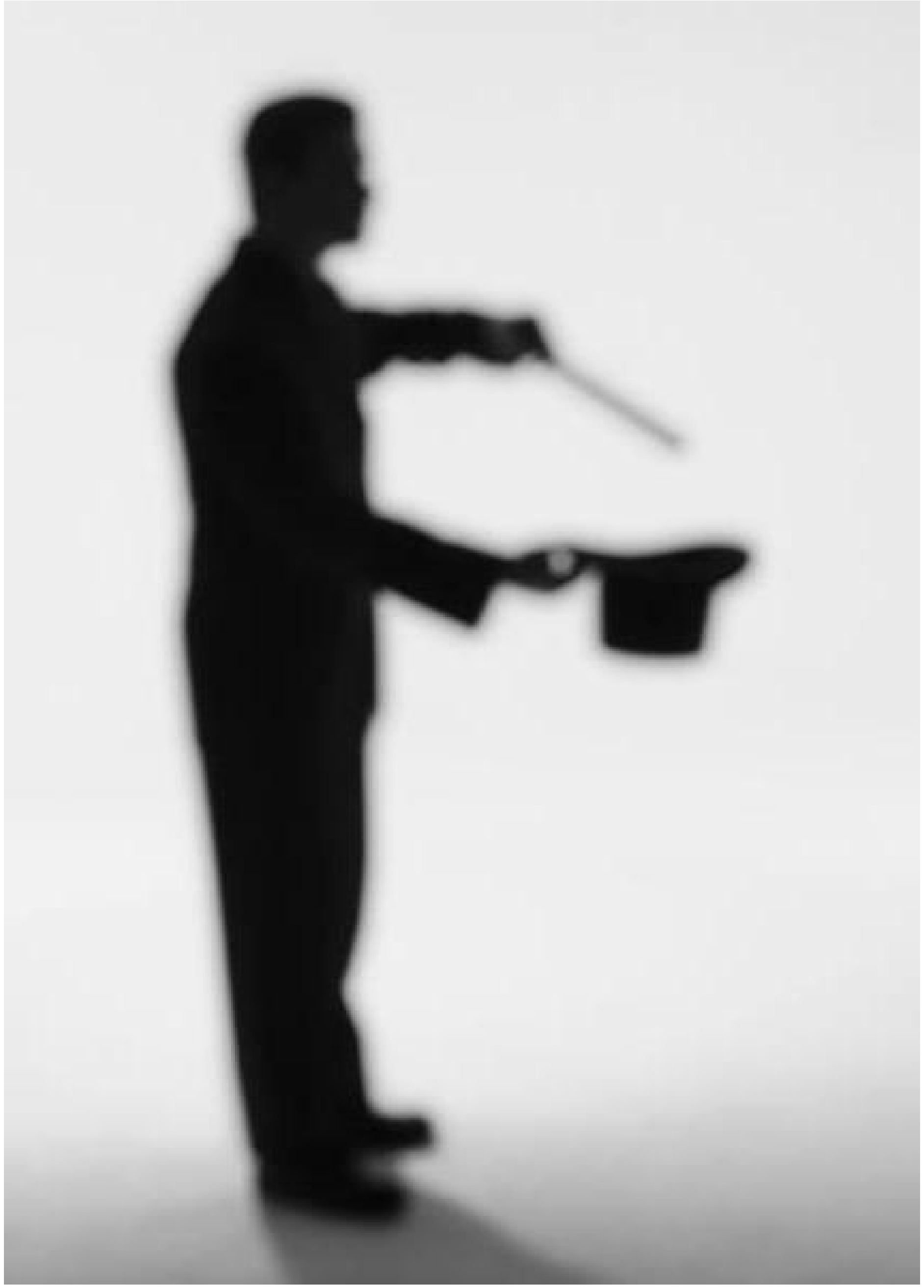
In fact, all of the generating functions shown are special cases of more refined generating functions for $C_k \wr S_n$ where we keep track of more statistics. For $\Upsilon \subseteq C_k \wr S_j$, we shall consider generating functions of the form

$$D_k^{\Upsilon}(x,p,q,r,t) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} x^{\Upsilon\text{-mch}((\sigma,w))} \tag{1}$$

where $||w|| = ||w_1 \dots w_n|| = w_1 + \cdots + w_n$.

$$D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} x^{\mathrm{ris}((\sigma,w))} = \frac{1-x}{1-x+\sum_{n\geq 1} \frac{p^{\binom{n}{2}}((x-1)t)^n}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r} \tag{2}$$

which reduce to the previous functions when we set $p=q=r=1$.



Let $Av_{n,k}^{\Upsilon}$:= the number of elements of $S_n \wr C_k$ which avoid a collection of patterns Υ.

Let $A_k^{\Upsilon}(t)$:= egf for the number of elements of $S_n \wr C_k$ which avoid a collection of patterns Υ.

Thm. (KNRR)

$$\begin{aligned}
A_k^{\Upsilon_r}(t) &= \frac{1}{1+\sum_{n\geq 1}\frac{(-t)^n}{n!}\binom{n+k-1}{n}} \\
A_k^{\Upsilon_w}(t) &= \frac{1}{1+k(e^{-t}-1)} \\
A_k^{\Upsilon_s}(t) &= \frac{1}{1+\sum_{n\geq 1}\frac{(-t)^n}{n!}\binom{k}{n}} \\
A_k^{\Upsilon_d}(t) &= \frac{k-1}{k-1+k(e^{t(1-k)}-1)}
\end{aligned}$$

Numbers involved and bijective questions

$$\Upsilon_{\mathbf{r}} = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$$

Table 1: $Av^{\Upsilon_{\mathbf{r}}}_{n,k}$ for $k, n \leq 5$.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$k = 2$	1	2	5	16	65	326
$k = 3$	1	3	12	64	441	3771
$k = 4$	1	4	22	164	1589	19136
$k = 5$	1	5	35	335	4180	64876

$\Upsilon_{\mathbf{r}} = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$

Table 2: $Av_{n,k}^{\Upsilon_{\mathbf{r}}}$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	5	16	65	326
$k=3$	1	3	12	64	441	3771
$k=4$	1	4	22	164	1589	19136
$k=5$	1	5	35	335	4180	64876

$Av_{2,k}^{\Upsilon_{\mathbf{r}}}$ = pentagonal numbers (A000326), (also the number of permutations in S_{n+1} avoiding 1-2-3])

$Av_{3,k}^{\Upsilon_{\mathbf{r}}}$ = the structured octagonal anti-prism numbers (A100184).

$Av_{n,2}^{\Upsilon_{\mathbf{r}}}$ = total $\sharp$ of arrangements of all subsets of $[n]$ (A000522).

$$
\begin{aligned}
A_{1,k}^{\Upsilon_{\mathbf{r}}} &= k \\
A_{2,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{2}k(3k-1) \\
A_{3,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{6}k(19k^2-15k+2) \\
A_{4,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{24}k(211k^3-270k^2+89k-6) \\
A_{5,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{120}k(3651k^4-6490k^3+3585k^2-650k+24)
\end{aligned}
$$

$$\begin{aligned}
A_{1,k}^{\Upsilon_{\mathbf{r}}} &= k \\
A_{2,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{2}k(3k-1) \\
A_{3,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{6}k(19k^2-15k+2) \\
A_{4,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{24}k(211k^3-270k^2+89k-6) \\
A_{5,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{120}k(3651k^4-6490k^3+3585k^2-650k+24)
\end{aligned}$$

Conj. For $n \geq 1$ and $k \geq 2$, $A_{n,k}^{\Upsilon_{\mathbf{r}}}=\frac{k}{n!}P_n(k)$ where $P_n(k)$ is a polynomial of degree $n-1$ whose leading coefficient is positive and such that signs of the remaining coefficients alternate.

$$\Upsilon_{\mathbf{w}} = \{(1\ 2, 0\ 0)\}$$

Table 3: $|Av_{n,k}(\Upsilon_{\mathbf{w}})|$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	6	26	150	1082
$k=3$	1	3	15	111	1095	13503
$k=4$	1	4	28	292	4060	70564
$k=5$	1	5	45	605	10845	243005

$\Upsilon_{\mathbf{w}} = \{(1\ 2, 0\ 0)\}$

Table 4: $Av_{n,k}^{\Upsilon_{\mathbf{w}}}$ for $k, n \leq 5$.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$k = 2$	1	2	6	26	150	1082
$k = 3$	1	3	15	111	1095	13503
$k = 4$	1	4	28	292	4060	70564
$k = 5$	1	5	45	605	10845	243005

$Av_{2,k}^{\Upsilon_{\mathbf{w}}}$ = hexagonal numbers (A000384).

$Av_{n,2}^{\Upsilon_{\mathbf{w}}}$ = number of necklaces on set of labeled beads (A000629).

$$\begin{aligned}
A_{0,k}^{\Upsilon_{\mathbf{w}}} &= 1 \\
A_{1,k}^{\Upsilon_{\mathbf{w}}} &= k \\
A_{2,k}^{\Upsilon_{\mathbf{w}}} &= \frac{1}{2}k(2k-1) \\
A_{3,k}^{\Upsilon_{\mathbf{w}}} &= \frac{1}{6}k(6k^2-6k+1) \\
A_{4,k}^{\Upsilon_{\mathbf{w}}} &= \frac{1}{24}k(24k^3-36k^2+14k-1) \\
A_{5,k}^{\Upsilon_{\mathbf{w}}} &= \frac{1}{120}k(120k^4-240k^3+150k^2-30k+1)
\end{aligned}$$

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A_{3,k}^{\Upsilon_{\mathbf{w}}} &= \frac{1}{6}k(6k^2-6k+1) \\
A_{4,k}^{\Upsilon_{\mathbf{w}}} &= \frac{1}{24}k(24k^3-36k^2+14k-1) \\
A_{5,k}^{\Upsilon_{\mathbf{w}}} &= \frac{1}{120}k(120k^4-240k^3+150k^2-30k+1)
\end{aligned}$$

Conj. For $n \geq 1$ and $k \geq 2$, $A_{n,k}^{\Upsilon_{\mathbf{w}}} = \frac{k}{n!}Q_n(k)$, where the coefficients in $Q_n(k)$ count the number of ordered set partitions up to sign.

$$\Upsilon_{\mathbf{s}} = \{(1\ 2, 0\ 1)\}$$

Table 5: $Av_{n,k}^{\Upsilon_{\mathbf{s}}}$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	7	36	246	2100
$k=3$	1	3	15	109	1050	12630
$k=4$	1	4	26	244	3031	47000
$k=5$	1	5	40	460	6995	132751

$\Upsilon_{\mathbf{s}} = \{(1\ 2, 0\ 1)\}$

Table 6: $Av_{n,k}^{\Upsilon_{\mathbf{s}}}$ for $k, n \leq 5$.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$k = 2$	1	2	7	36	246	2100
$k = 3$	1	3	15	109	1050	12630
$k = 4$	1	4	26	244	3031	47000
$k = 5$	1	5	40	460	6995	132751

$Av_{2,k}^{\Upsilon_{\mathbf{s}}}$ = second pentagonal numbers (A005449).

$$\begin{aligned}
A^{\Upsilon_{\mathsf{s}}}_{1,k} &= k\\
A^{\Upsilon_{\mathsf{s}}}_{2,k} &= \frac{1}{2}k(3k+1)\\
A^{\Upsilon_{\mathsf{s}}}_{3,k} &= \frac{1}{6}k(19k^2+15k+2)\\
A^{\Upsilon_{\mathsf{s}}}_{4,k} &= \frac{1}{24}k(211k^3+270k^2+89k+6)\\
A^{\Upsilon_{\mathsf{s}}}_{5,k} &= \frac{1}{120}k(3651k^4+6490k^3+3585k^2+650k+2)
\end{aligned}$$

$$\begin{aligned}
A_{1,k}^{\Upsilon_{\mathsf{s}}} &= k \\
A_{2,k}^{\Upsilon_{\mathsf{s}}} &= \frac{1}{2}k(3k+1) \\
A_{3,k}^{\Upsilon_{\mathsf{s}}} &= \frac{1}{6}k(19k^2+15k+2) \\
A_{4,k}^{\Upsilon_{\mathsf{s}}} &= \frac{1}{24}k(211k^3+270k^2+89k+6) \\
A_{5,k}^{\Upsilon_{\mathsf{s}}} &= \frac{1}{120}k(3651k^4+6490k^3+3585k^2+650k+2)
\end{aligned}$$

Conj. For $n \geq 1$ and $k \geq 2$, $A_{n,k}^{\Upsilon_{\mathsf{s}}} = \frac{1}{n!}kR_n(k)$ where $R_n(k)$ is a polynomial of degree $n-1$ with positive coefficients, and match the coefficients of $P_n(k)$ up to sign.

$$\Upsilon_{\mathbf{d}} = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}.$$

Table 7: $Av_{n,k}^{\Upsilon_{\mathbf{d}}}$ for $k, n \le 5$.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
k=2	1	2	6	26	150	1082
k=3	1	3	12	66	480	4368
k=4	1	4	20	132	1140	12324
k=5	1	5	30	230	2280	28280

$\Upsilon_{\mathbf{d}} = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$.

Table 8: $Av_{n,k}^{\Upsilon_{\mathbf{d}}}$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
k=2	1	2	6	26	150	1082
k=3	1	3	12	66	480	4368
k=4	1	4	20	132	1140	12324
k=5	1	5	30	230	2280	28280

$Av_{2,k}^{\Upsilon_{\mathbf{d}}} = k(k+1)$ =number of necklaces on set of labeled beads (A000629).

$$\begin{aligned}
A^{\Upsilon_{\mathbf{d}}}_{1,k} &= k \\
A^{\Upsilon_{\mathbf{d}}}_{2,k} &= \frac{1}{2}k(k+1) \\
A^{\Upsilon_{\mathbf{d}}}_{3,k} &= \frac{1}{6}k(k^2+4k+1) \\
A^{\Upsilon_{\mathbf{d}}}_{4,k} &= \frac{1}{24}k(k^3+11k^2+11k+1) \\
A^{\Upsilon_{\mathbf{d}}}_{5,k} &= \frac{1}{120}k(k^4+26k^3+66k^2+26k+1)
\end{aligned}$$

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A^{\Upsilon_{\mathbf{d}}}_{1,k} &= k \\
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A^{\Upsilon_{\mathbf{d}}}_{4,k} &= \frac{1}{24}k(k^3+11k^2+11k+1) \\
A^{\Upsilon_{\mathbf{d}}}_{5,k} &= \frac{1}{120}k(k^4+26k^3+66k^2+26k+1)
\end{aligned}$$

Thm. For $n \geq 1$ and $k \geq 2$, $A^{\Upsilon_{\mathbf{d}}}_{n,k} = \frac{1}{n!}k(E_n(k))$, where E_n is the n^{th} Eulerian polynomial.

$$\begin{aligned}
A^{\Upsilon_{\mathbf{d}}}_{1,k} &= k \\
A^{\Upsilon_{\mathbf{d}}}_{2,k} &= \frac{1}{2}k(k+1) \\
A^{\Upsilon_{\mathbf{d}}}_{3,k} &= \frac{1}{6}k(k^2+4k+1) \\
A^{\Upsilon_{\mathbf{d}}}_{4,k} &= \frac{1}{24}k(k^3+11k^2+11k+1) \\
A^{\Upsilon_{\mathbf{d}}}_{5,k} &= \frac{1}{120}k(k^4+26k^3+66k^2+26k+1)
\end{aligned}$$

Thm. For $n \geq 1$ and $k \geq 2$, $A^{\Upsilon_{\mathbf{d}}}_{n,k} = \frac{1}{n!}k(E_n(k))$, where E_n is the n^{th} Eulerian polynomial.

(Probably. Proof sketched at dinner at I 7 Peccati at 9PM last night.)

Distribution of non-overlapping Υ-bi-matches

Using techniques developed by Kitaev in 2007, we can use A_k^{Υ} to find the egf for the distribution of $\Upsilon - nlap$.

Thm. For all $\Upsilon \subseteq C_k \wr S_j$ and $k \geq 2$,

$$N_k^{\Upsilon}(t) = \frac{A_k^{\Upsilon}(t)}{1 - x(1 + (k-1)A_k^{\Upsilon}(t))}.$$

Preview of the Distributed Case

Simultaneous avoidance of the patterns (1-2,0 0) and (1-2,0 1)

Theorem 0.1. *The number of permutations in* $C_2 \wr S_n$ *simultaneously avoiding (1-2,0 0) and (1-2,0 1) is given by the* $(n+1)$*-st Catalan number* $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$.

Pf. Notice the underlying permutation avoids $1-2-3$, since with only two labels, 0 and 1, we cannot avoid (1-2,0 0) and (1-2,0 1).

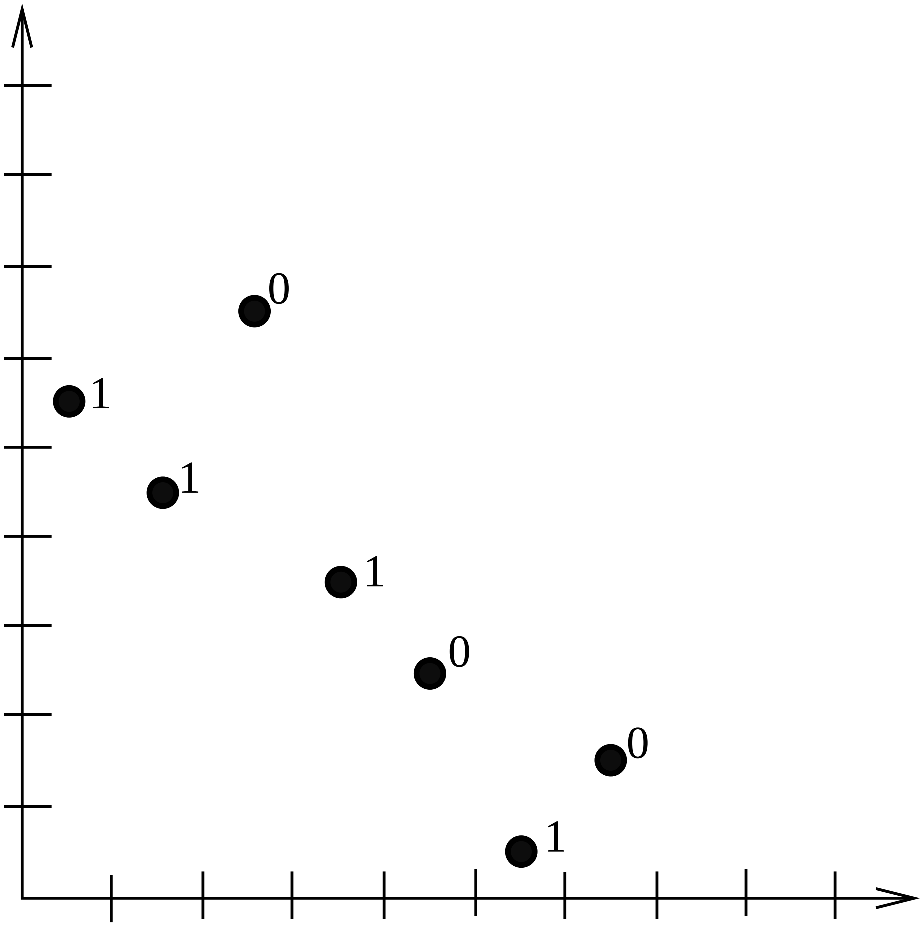

Figure 1: Matrix representation of $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$.

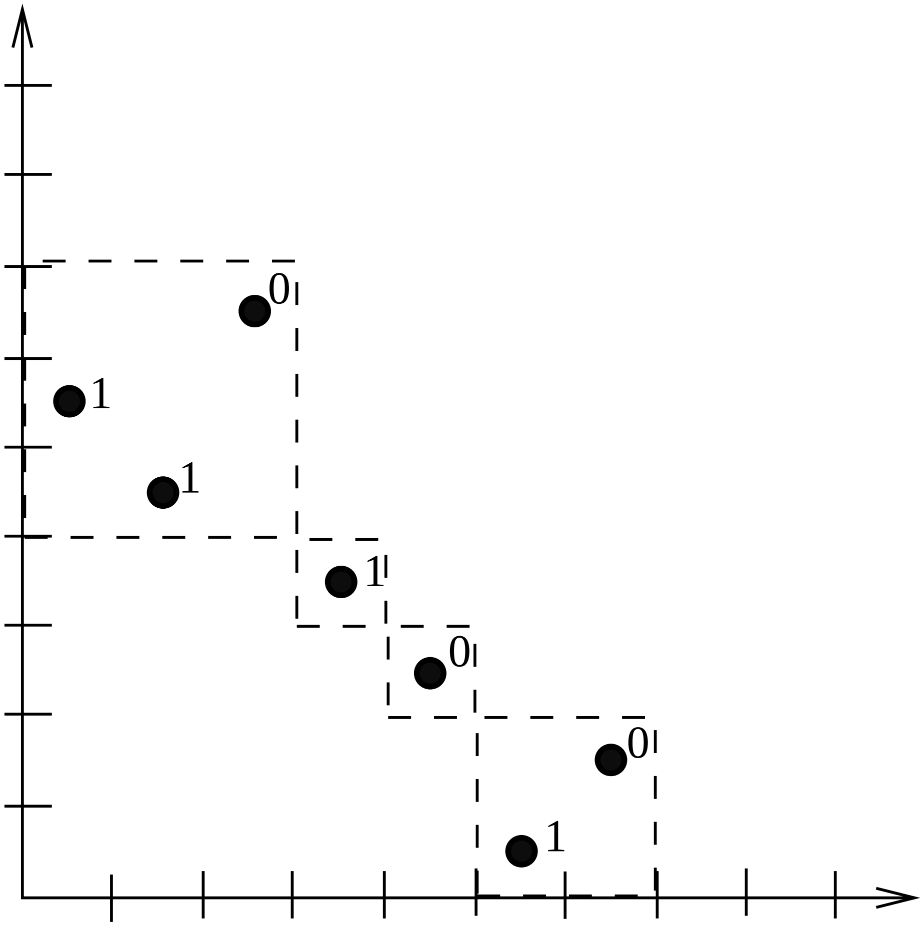

Figure 2: Matrix representation of $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$ with reverse irreducible blocks outlined.

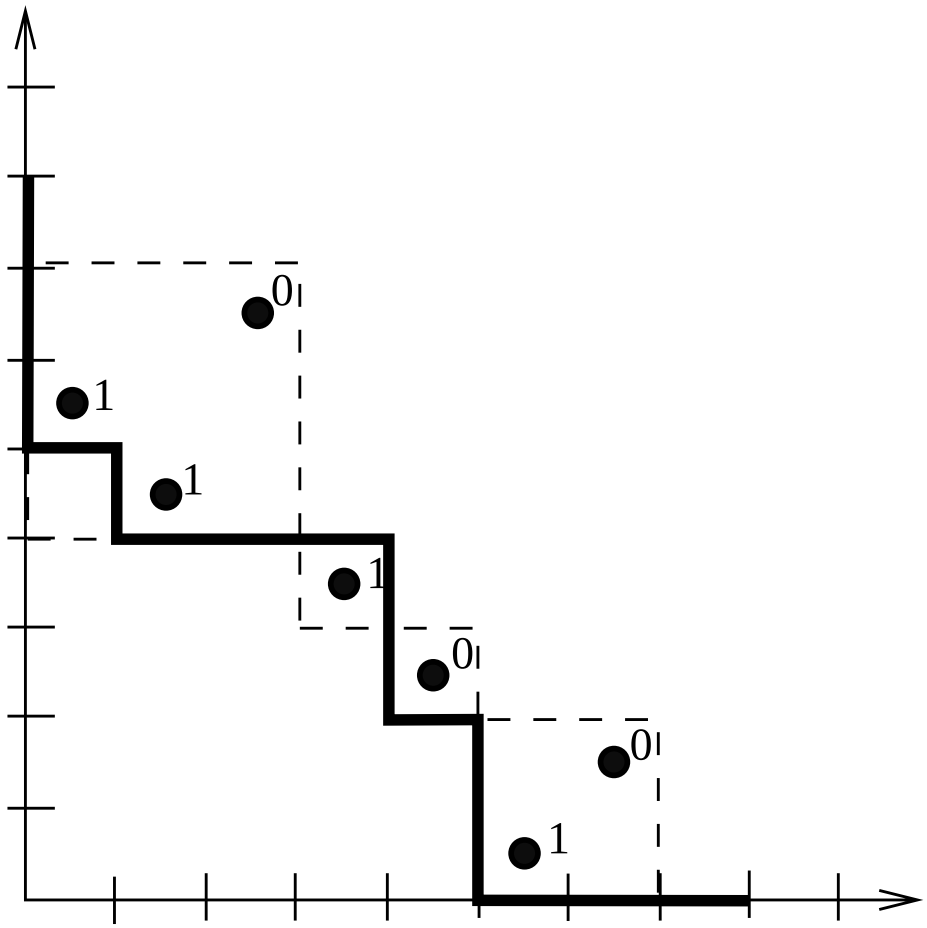

Figure 3: The Dyck path corresponding to $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$ added to Figure 2.

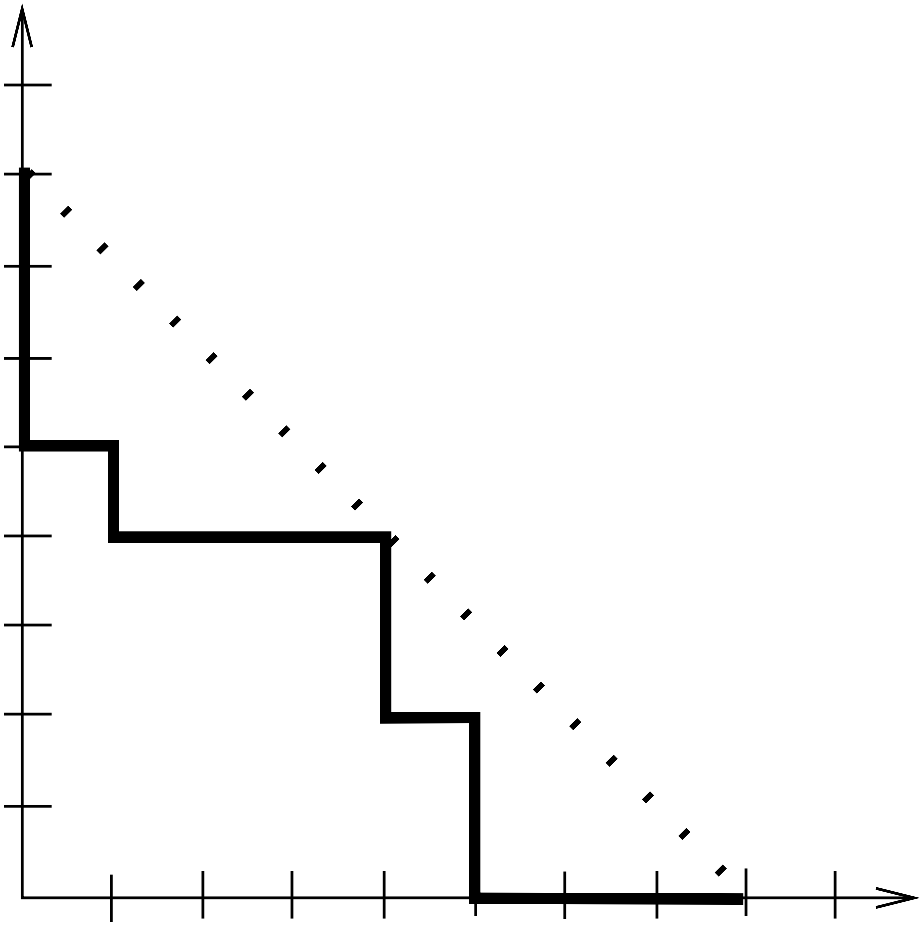

Figure 4: The Dyck path corresponding to $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$. Note the path only touches the line $y = -x + n + 1$ after a singleton colored 1.

Working on now:

- Collections of patterns of length ≥ 3
- k-tuples of words
- Distributed patterns of longer lengths

Untouched:

- Patterns analogous to 31-2
- Barred patterns

Thank you!