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Some permutations with forbidden subsequences and their inversion number

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Abstract

A permutation π avoids the subpattern τ iff π has no subsequence having all the same pairwise comparisons as τ, and we write $\pi \in S(\tau)$. We examine the classes of permutations, $S(321)$, $S(321, 3\bar{1}42)$ and $S(4231, 4132)$, enumerated, respectively by the famous Catalan, Motzkin and Schröder number sequences. We determine their generating functions according to their length, number of active sites and inversion number. We also find the average inversion number for each class. Finally, we describe some bijections between these classes of permutations and some classes of parallelogram polyominoes, from which we deduce some relations between the parameters of Motzkin and Schröder permutations. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Permutations with forbidden subsequences have been widely studied by many authors. A survey of the methods and results involved can be found in Gire's [5], Guibert's [6] and West's [9], Ph.D. theses. Knuth [7, p.238] proved that the first class, $S(321)$, is enumerated by Catalan numbers. Gire [5] and West [10] enumerated the second class, $S(321, 3\bar{1}42)$, by means of generating trees. These trees allow us to deduce a recursive construction for every class. By translating each construction into its corresponding generating function according to the length, number of active sites and inversion number, we obtain a functional equation solvable by Bousquet–Mélou's lemma [2].

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Section 2 of this paper contains some definitions on permutations having forbidden subsequences and, in Sections 3–5, we determine the generating functions of $S(321)$, $S(321, 3\bar{1}42)$ and $S(4231, 4132)$ permutations according to their length, number of active sites and inversion number. In Section 6, we describe some bijections between:

- $S_n(321)$ and parallelogram polyominoes having half-perimeter equal to $n+1$,
- $S_n(321, 3\bar{1}42)$ and steep parallelogram polyominoes having half-perimeter equal to $n+2$,
- $S_n(4231, 4132)$ and steep parallelogram polyominoes having height equal to $n+1$.

From the last two bijections we get some relations between the parameters of Motzkin and Schröder permutations.

2. Notations and definitions

A permutation $\pi = \pi(1)\pi(2)\ldots\pi(n)$ on $[n] = \{1, 2, \ldots, n\}$ is a bijection between $[n]$ and $[n]$. Let S_n be the set of permutations on $[n]$.

Definition 2.1. A permutation $\pi \in S_n$ contains a subsequence of type $\tau \in S_k$ if a sequence of indexes $1 \leqslant i_{\tau(1)} < i_{\tau(2)} < \cdots < i_{\tau(k)} \leqslant n$ exists such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. We denote the set of permutations of S_n not containing subsequences of type τ by $S_n(\tau)$.

Definition 2.2. A *barred* permutation $\bar{\tau}$ of $[k]$ is a permutation of S_k having a bar over one of its elements. Let τ be the permutation on $[k]$ obtained by unbarring $\bar{\tau}$, and $\hat{\tau}$ the permutation of $[k-1]$ made up of the $k-1$ unbarred elements of $\bar{\tau}$, rearranged to be a permutation on $[k-1]$.

Example 2.1. If $\bar{\tau} = 41\bar{3}52$ on [5], we have $\tau = 41352$ and $\hat{\tau} = 3142$.

Definition 2.3. A permutation $\pi \in S_n$ contains a type $\bar{\tau}$ subsequence if π contains a type $\hat{\tau}$ subsequence that, in turn, is not a type τ subsequence. We denote by $S_n(\bar{\tau})$ the set of permutations of S_n not containing type $\bar{\tau}$ subsequences.

Example 2.2. Let $\bar{\tau} = 41\bar{3}52$. The permutation $\pi = 6145732$ belongs to $S_7(\bar{\tau})$ because all its subsequences of type 3142: $\pi(1), \pi(2), \pi(5), \pi(6) = 6173$, and $\pi(1), \pi(2), \pi(5), \pi(7) = 6172$ are subsequences of $\pi(1), \pi(2), \pi(3), \pi(5), \pi(6) = 61473$ and $\pi(1), \pi(2), \pi(3), \pi(5), \pi(7) = 61472$, which are of type $\tau = 41352$.

Definition 2.4. Let $\{\tau_1, \ldots, \tau_p\}$ be a set of barred or unbarred permutations. We denote the set $S_n(\tau_1) \cap \cdots \cap S_n(\tau_p)$ by $S_n(\tau_1, \ldots, \tau_p)$.

We call the family $F=\{\tau_1,\dots,\tau_p\}$ a family of forbidden subsequences, the set $S_n(F)$ a family of permutations avoiding the subsequences in F and $S(F)=\bigcup_{n\geqslant 1} S_n(F)$ a class of permutations avoiding the subsequences in F.

Definition 2.5. A *site* for a permutation $\pi=\pi(1)\pi(2)\dots\pi(n)\in S_n$ is a position lying between two consecutive elements $\pi(i)$ and $\pi(i+1)$ for $i\in[1,n-1]$, or to the left of $\pi(1)$, or to the right of $\pi(n)$.

Definition 2.6. For every family $F=\{\tau_1,\dots,\tau_p\}$, a site of a permutation $\pi\in S_n(F)$ is *active* if the insertion of $n+1$ in that site gives a permutation belonging to the set $S_{n+1}(F)$; otherwise it is said to be *inactive*.

Definition 2.7. Let $\pi\in S_n$. The pair (i,j), with $i<j$, is an *inversion* if $\pi(i)>\pi(j)$. An element $\pi(i)$ is a right minimum if $\pi(i)<\pi(j)$, $\forall j\in[i+1,n]$.

A labelled tree is said to be a *generating tree* if it has the following property: any two vertices v and w having the same label have the same number of sons, and the multisets $M(v)$ and $M(w)$ formed by the labels of their sons are the same (see Fig. 1 for an example). Therefore, any generating tree can be built up by a *recursive rewriting rule* consisting of

1. the label of the root
2. a set of *succession rules* explaining how to derive the number of sons and their labels when the label of the father is known.

In the next sections, we examine the following permutation classes: $S(321)$, $S(321, 3\bar{1}42)$ and $S(4231, 4132)$, enumerated by the famous sequences of Catalan, Motzkin and Schröder numbers. For each class, we establish the active sites' position (Lemmas 3.1, 4.1 and 5.1) and the recursive rewriting rule (Lemmas 3.2, 4.2 and 5.2), by means of which we deduce the functional equations verified by the generating function of each class. We find the generating functions according to the following parameters: permutation length, number of active sites and inversion number.

3. Catalan permutations

Our enumerative results on Catalan permutations use the following two lemmas which were proved by West [10]:

Lemma 3.1. *Let π be a permutation of $S_n(321)$. If s is an active site of π, then each site on its right is also active.*

Lemma 3.2. *Let k be the number of active sites of $\pi\in S(321)$, the generating tree of Catalan permutations is isomorphic to the tree obtained by applying the following*

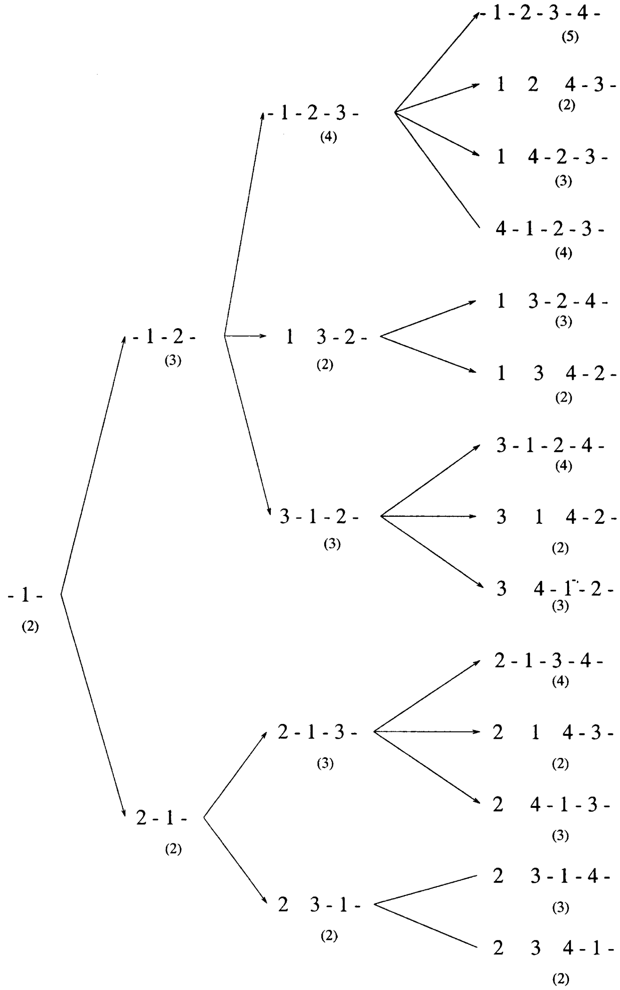

Fig. 1. The generating tree for Catalan permutations.

recursive rewriting rule:
Root : (2)
Rule : $(k) \to (k+1)(2)\dots(k-1)(k)$.
(see Fig. 1).

Let $\pi \in S(321)$, we denote: its length by $n(\pi)$, the number of its active sites by $a(\pi)$, its inversion number by $i(\pi)$. The generating function of $S(321)$ permutations according to the above-mentioned parameters is the following:

$$C(s,x,q) = \sum_{\pi \in S(321)} s^{a(\pi)} x^{n(\pi)} q^{i(\pi)}.$$

Let us number the active sites of $\pi=\pi(1)\ldots\pi(n)$ from right to left in increasing order.

$$\Pi(1)\ \Pi(2)\ \ldots\ \Pi(n-a+1) \underset{a}{\mapsto} \Pi(n-a+2) \underset{a-1}{\mapsto} \cdots \underset{3}{\mapsto} \Pi(n-1) \underset{2}{\mapsto} \Pi(n) \underset{1}{\mapsto}$$

Let π' be the permutation obtained from π by putting $n+1$ into the kth active site; the parameters change as follows:

- if $k=1$ then $n(\pi')=n(\pi)+1,\ a(\pi')=a(\pi)+1,\ i(\pi')=i(\pi)$;
- if $2\leqslant k\leqslant a(\pi)$ then $n(\pi')=n(\pi)+1,\ a(\pi')=k,\ i(\pi')=i(\pi)+k-1$.

The translation of this construction in terms of generating functions gives the following expression for the generating function of elements of $S_n(321)$ of length at least 2.

$$\frac{x}{q}\sum_{\pi\in S(321)} x^{n(\pi)}q^{i(\pi)}\sum_{k=2}^{a(\pi)}(sq)^k + sx\sum_{\pi\in S(321)} s^{a(\pi)}x^{n(\pi)}q^{i(\pi)}$$
$$=\frac{x}{q}\sum_{\pi\in S(321)} x^{n(\pi)}q^{i(\pi)}\frac{s^2q^2-(sq)^{a(\pi)+1}}{1-sq} + sxC(s,x,q)$$

and, if we assume that s^2x is the weight of the permutation having length one, we have:

Proposition 3.3. *The generating function of* $S(321)$ *permutations*, $C(s,x,q)$, *satisfies the following functional equation*:

$$C(s,x,q)=\frac{s^2x}{1-sx}+\frac{s^2xq}{(1-sx)(1-sq)}C(1,x,q)-\frac{sx}{(1-sx)(1-sq)}C(sq,x,q).$$

By Bousquet–Mélou's lemma [2], we have:

Theorem 3.4. *The generating function* $C(s,x,q)$ *is given by*

$$C(s,x,q)=\frac{J_1(s)J_0(1)-J_1(1)J_0(s)+J_1(1)}{J_0(1)}, \qquad (3.0.1)$$

where

$$J_0(s,x,q)=1-\sum_{n\geqslant 0}\frac{(-1)^n s^{n+2}x^{n+1}q^{[(n+1)(n+2)]/2}}{(xs,q)_{n+1}(sq,q)_{n+1}},$$

$$J_1(s,x,q)=\sum_{n\geqslant 0}\frac{(-1)^n s^{n+2}x^{n+1}q^{[n(n+3)]/2}}{(xs,q)_{n+1}(sq,q)_{n}};$$

and $(a,q)_n=\prod_{k=0}^{n-1}(1-aq^k)$.

In the sequel, we denote the generical function $F(1,x,y,q)$ by $F(x,y,q)$.

By putting $s = 1$, the expression 3.0.1 becomes simpler:

$$C(x,q) = \frac{\sum_{n\geqslant 0}(-1)^n x^{n+1} q^{[n(n+3)]/2}/(x,q)_{n+1}(q,q)_n}{\sum_{n\geqslant 0}(-1)^n x^n q^{[n(n+1)]/2}/(x,q)_n(q,q)_n};$$

moreover, by means of some computations, we obtain:

Lemma 3.5. *The functions* $J_0(x,q)$ *and* $J_1(x,q)$ *satisfy the following equations*:

$$xJ_0(xq,q) = (1-x)J_1(x,q),$$

$$xqJ_1(xq,q) = (1-x)((J_0(xq,q) - J_0(x,q)) - qJ_1(x,q)).$$

From Theorem 3.4 and Lemma 3.5, we deduce that

$$xqC(x,q)C(xq,q) = (1-x-xq)C(x,q) - x. \tag{3.0.2}$$

If we put $q = 1$:

$$C(x) = \frac{1-2x-\sqrt{1-4x}}{2x} = \sum_{n\geqslant 0} C_n x^n - 1.$$

Therefore, we find the result of Knuth [7, p. 238]:

Proposition 3.6. *The generating function* $C(x)$ *is the Catalan numbers generating function and so* $|S_n(321)| = C_n = [1/(n+1)]\binom{2n}{n}$.

Let us now determine the average inversion number I_n of Catalan permutations of length n:

$$I_n = \frac{[x^n](\partial/\partial q)C(x,q)_{|q=1}}{[x^n]C(x)}.$$

By setting $I(x) = (\partial/\partial q)C(x,q)_{|q=1}$, from Eq. (3.0.2), we obtain

$$xC^2(x) + xI(x)C(x) + xC(x)\frac{\partial}{\partial q}C(xq,q)_{|q=1} = -xC(x) + (1-2x)I(x).$$

Hence,

$$I(x) = \frac{1-\sqrt{1-4x}-2x}{2(1-4x)}$$

and so:

Proposition 3.7. *The average inversion number of Catalan permutations of length* n *is*

$$I_n = \frac{\sqrt{\pi}}{4}n^{3/2} + o(n).$$

Notice that a random permutation of length n has $n(n-1)/4$ inversions.

4. Motzkin permutations

In order to extend the enumerative results obtained by Gire and West on Motzkin permutations let us recall the following two lemmas [10].

Lemma 4.1. *Let π be a permutation of $S_n(321, 3\bar{1}42)$. If s is an active site of π, then each site to its right is active too.*

Lemma 4.2. *Let k be the number of active sites of $\pi \in S(321, 3\bar{1}42)$, then the generating tree of Motzkin permutations is isomorphic to the tree obtained by applying the following recursive rewriting rule*:

$$\mathit{Root}: (2),$$
$$\mathit{Rule}: (k) \to (k+1)(1)\dots(k-2)(k-1).$$

In this case, the parameter which counts the permutation length is divided into two parts: if $\pi \in S(321, 3\bar{1}42)$, we denote: its length by $m(\pi) + l(\pi)$, where $l(\pi)$ is the number of its right minima. Moreover, we denote the number of active sites of π by $a(\pi)$ and its inversion number by $i(\pi)$. The generating function of $S(321, 3\bar{1}42)$ permutations according to the above-mentioned parameters is the following:

$$M(s,x,y,q) = \sum_{\pi \in S(321,3\bar{1}42)} s^{a(\pi)} x^{m(\pi)} y^{l(\pi)} q^{i(\pi)}.$$

Let $\pi \in S_n(321, 3\bar{1}42)$. We number the active sites of π from right to left in increasing order. Let π' be the permutation obtained from π by putting $n+1$ into the kth active site; the parameters change as follows:

- if $k = 1$ then $m(\pi') = m(\pi)$, $l(\pi') = l(\pi) + 1$, $a(\pi') = a(\pi) + 1$, $i(\pi') = i(\pi)$;
- if $2 \leqslant k \leqslant a(\pi)$ then $m(\pi') = m(\pi) + 1$, $l(\pi') = l(\pi)$, $a(\pi') = k - 1$, $i(\pi') = i(\pi) + k - 1$.

If we assume that $s^2 y$ is the weight of the permutation of length one, by proceeding as in the first example, we obtain

$$M(s,x,y,q) = \frac{s^2 y}{1 - sy} + \frac{sxq}{(1 - sq)(1 - sy)} M(1,x,y,q) - \frac{x}{(1 - sq)(1 - sy)} M(sq,x,y,q).$$

Theorem 4.3. *The generating function $M(s,x,y,q)$ is given by*

$$M(s,x,y,q) = \frac{J_1(s)J_0(1) - J_1(1)J_0(s) + J_1(1)}{J_0(1)},$$

where

$$J_0(s,x,y,q) = 1 - sxq\sum_{n\geqslant 0}\frac{(-1)^n x^n q^n}{(sq,q)_{n+1}(sy,q)_{n+1}},$$

$$J_1(s,x,y,q) = s^2 y\sum_{n\geqslant 0}\frac{(-1)^n x^n q^{2n}}{(sq,q)_n(sy,q)_{n+1}}.$$

By setting $s=1$, from Theorem 4.3 we obtain

$$M(x,y,q) = \frac{y\sum_{n\geqslant 0}[(-1)^n x^n q^{2n}]/[(q,q)_n(y,q)_{n+1}]}{\sum_{n\geqslant 0}[(-1)^n x^n q^n]/[(q,q)_n(y,q)_n]};$$

moreover, it yields the following lemma.

Lemma 4.4. *The functions $J_0(x,y,q)$ and $J_1(x,y,q)$ satisfy the following equations*:

$$J_1(x,y,q) - yJ_1(xq,y,q) = yJ_0(xq,y,q),$$

$$J_0(xq,y,q) - J_0(x,y,q) = xq(J_0(x,y,q) + J_1(x,y,q)).$$

We now obtain

$$xyqM(x,y,q)M(xq,y,q) + y(1+xq)M(xq,y,q) = (1-xyq)M(x,y,q) - y(1+xq)$$

and, for $q=1, y=x$:

$$M(1,x,x,1) = \frac{1-x-2x^2-\sqrt{-3x^2-2x+1}}{2x^2} = \sum_{n\geqslant 0} M_n x^n - 1.$$

Therefore, we obtain West's result [10].

Proposition 4.5. *The generating function $M(x)$ is the Motzkin numbers generating function and so $|S_n(321,3\bar{1}42)| = M_n = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k$, where C_k is the kth Catalan number.*

By putting $q=1$ and $x=1$, we obtain the Schröder number generating function:

$$M(1,1,y,1) = \frac{1-y-\sqrt{1-6y+y^2}}{2y} - 1.$$

Therefore, we deduce the following nice result:

Proposition 4.6. *The number of Motzkin permutations having n right minima is equal to $(n+1)$th Schröder number, that is $R_n = \sum_{k=0}^{n}\binom{n+k}{2k}C_k$, where C_k is the kth Catalan number.*

In the last section, we give a combinatorial proof of this proposition. By proceeding as in the previous section, we can deduce:

Proposition 4.7. *The average inversion number of Motzkin permutations of length n is*

$$I_n = \frac{1}{3}\sqrt{\frac{\pi}{3}}n^{3/2} + o(n).$$

5. Schröder permutations

We have similar results as for Catalan and Motzkin permutations:

Lemma 5.1. *Let π be a permutation of $S_n(4231, 4132)$. If s is an active site of π, then each site to its right is also active.*

Proof. Let s be an active site of π and let t be a site to its right. If the insertion of $n+1$ in t produces a subsequence $n+1, i, j, k$ of 4231 type or $n+1, k, j, i$ of 4132 type, then the insertion of $n+1$ in s produces the same subsequence. Therefore, s is not active. □

Moreover, in [10] West proves:

Lemma 5.2. *Let k be the number of active sites of $\pi \in S(4231, 4132)$, then the generating tree of Schröder permutations is isomorphic to the tree obtained by applying the following recursive rewriting rule*:

$$\begin{aligned} &Root: (2) \\ &Rule: (k) \rightarrow (k+1)(3)\dots(k)(k+1). \end{aligned}$$

Let $\pi \in S(4231, 4132)$, we denote: its length by $n(\pi)$, the number of its active sites by $a(\pi)$, and its inversion number by $i(\pi)$. The generating function of $S(4231, 4132)$ permutations according to the above-mentioned parameters is the following:

$$S(s, x, q) = \sum_{\pi \in S(4231, 4132)} s^{a(\pi)} x^{n(\pi)} q^{i(\pi)}.$$

Let us number the active sites of $\pi = \pi(1)\dots\pi(n)$ from right to left in increasing order, and let π' be the permutation obtained from π by putting $n+1$ into the ith active site. The parameters change in the following way:

- if $i = 1$ then $n(\pi') = n(\pi) + 1$, $a(\pi') = a(\pi) + 1$, and $i(\pi') = i(\pi)$;
- if $2 \leqslant i \leqslant a(\pi)$ then $n(\pi') = n(\pi) + 1$, $a(\pi') = i + 1$, and $i(\pi') = i(\pi) + i - 1$.

The construction allows us to obtain the permutations $\pi \in S(4231,4132)$ such that $n(\pi) > 1$, and if we assume that s^2x is the weight of the permutation having length one, we obtain

$$S(s,x,q) = \frac{s^2x}{1-sx} + \frac{s^2x}{(1-sx)(1-sq)}[sqS(1,x,q) - S(sq,x,q)].$$

Theorem 5.3. *The generating function $S(s,x,q)$ is given by*:

$$S(s,x,q) = \frac{J_1(s)J_0(1) - J_1(1)J_0(s) + J_1(1)}{J_0(1)},$$

where

$$J_1(s) = \sum_{n\geqslant 0}(-1)^n \frac{x^{n+1}s^{2(n+1)}q^{n(n+1)}}{(sx,q)_{n+1}(sq,q)_n},$$

$$J_0(s) = 1 - \sum_{n\geqslant 0}(-1)^n \frac{x^{n+1}s^{2n+3}q^{(n+1)^2}}{(sx,q)_{n+1}(sq,q)_{n+1}}.$$

From Theorem 5.3, we immediately deduce the generating function of $S(4231,4132)$ permutations according to their length and number of active sites:

$$S(x,q) = \frac{\sum_{n\geqslant 0}(-1)^n(x^{n+1}q^{n(n+1)})/[(x,q)_{n+1}(q,q)_n]}{\sum_{n\geqslant 0}(-1)^n(x^nq^{n^2})/[(x,q)_n(q,q)_n]}.$$

After some computations we obtain:

Lemma 5.4. *The functions $J_0(x,q)$ and $J_1(x,q)$ satisfy the following equations*:

$$xJ_0(xq,q) = (1-x)J_1(x,q),$$
$$xJ_1(xq,q) = (1-x)[(1-x)J_1(x,q) - xJ_0(x,q)].$$

We can now write

$$S(x,q) = x + xS(x,q) + S(xq,q)S(x,q)$$

and, for $q = 1$:

$$S(1,x,1) = \frac{1-x-\sqrt{x^2-6x+1}}{2} = \sum_{n\geqslant 1} R_{n-1}x^n.$$

Consequently, we have [10]:

Theorem 5.5. *The generating function $S(x)$ is the Schröder number generating function*; *therefore*, $|S_n(4231,4132)| = R_{n-1} = \sum_{k=0}^{n-1}\binom{n+k-1}{2k}C_k$, *where C_k is the kth Catalan number*.

Moreover:

Proposition 5.6. *The generating function* $S(x,q)$ *satisfies*:

$$S(x,q) = \frac{x}{1 - x - xq/(1 - xq - \frac{xq^2}{\cdots})}.$$

By proceeding as in the previous section, we obtain:

Proposition 5.7. *The average inversion number of Schröder permutations of length n is*:

$$I_n = \sqrt{\frac{\pi}{4\sqrt{2}}} n^{3/2} + o(n).$$

6. Permutations and polyominoes

Before giving some definitions concerning polyominoes, we wish to clarify our aim in this section.

We examine some subclasses of polyominoes whose generating trees are in bijection with the permutation trees studied in the previous section and describe both the correspondence between their constructions and, when possible, some of their parameters.

Let us consider the plane $\Pi = Z \times Z$. A *cell* is a unit square in Π, a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to a translation. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite vertical (horizontal) strip of cells. The *area* of a polyomino is the number of its cells, the *height* (*width*) is the numbers of non empty rows (columns), respectively, and its *perimeter* is the length of its border. For a survey on polyomino enumeration, see [3,8].

We particularly deal with parallelogram polyominoes [1,4] (see Fig. 2).

Definition 6.1. A *parallelogram* polyomino is defined by two nonintersecting paths (their origin and extremity are the only intersection points) and only having north and east steps.

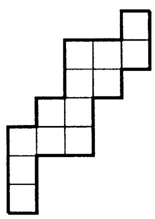

Fig. 2. A parallelogram polyomino.

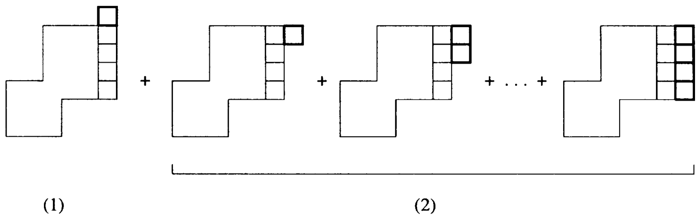

Fig. 3. The construction of parallelogram polyominoes.

6.1. Polyominoes and Catalan permutations

We denote the set of parallelogram polyominoes having half-perimeter n by P_n. Let us represent a generic parallelogram polyomino $P \in P_n$ of width m as follow:

$$P = \{(c_i, d_i),\ i = 1, \dots, m\},$$

where c_i is the height of the ith column and d_i is the number of cells in the ith column having no adjacent cell on their right (d_m is always equal to c_m).

The two operations that allow us to obtain all the polyominoes in P_{n+1} from the ones in P_n are the following:

1. add a cell onto the last column, then $P' = \{(c_1, d_1), (c_2, d_2), \dots, (c_m + 1, d_m + 1)\}$;
2. add a column having height $h \in [1, \dots, c_m]$ to the right of the last column; consequently:

$$P' = \{(c_1, d_1), (c_2, d_2), \dots, (c_m, d'_m), (c_{m+1}, d_{m+1}) \colon c_{m+1} = h,\ d'_m = d_m - h\},$$

(see Fig. 3).

Definition 6.2. A cell is said to be *right-joined* when it has an adjacent cell on its right.

We denote $n(P)$ the width of P, $l(P)$ its height, $a(P)$ its last column height, and $f(P)$, the number of right-joined cells of P (Fig. 4).

Let us observe how these parameters change when we apply the above two operations:

1. $n(P') = n(P),\ l(P') = l(P) + 1,\ a(P') = a(P) + 1,\ f(P') = f(P)$;
2. $n(P') = n(P) + 1,\ l(P') = l(P),\ a(P') = h,\ f(P') = f(P) + h$, for $h = 1, \dots, a(P)$.

The correspondence between these parameters and those of Catalan permutations is now clear: the half-perimeter is the length minus one, the height of the last column is the number of the active sites minus one, and the number of right-joined cells coincides with the inversion number.

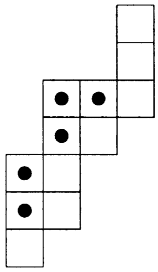

Fig. 4. The right-joined cells of a parallelogram polyomino.

6.2. Polyominoes, Motzkin and Schröder permutations

Definition 6.3. A *lower steep parallelogram* polyomino is a parallelogram polyomino whose south border has no consecutive horizontal steps.

In this section we show two bijections between:

- $S_n(321, 3\bar{1}42)$ and the steep parallelogram polyominoes having half-perimeter equal to $n+2$,
- $S_n(4231, 4132)$ and the steep parallelogram polyominoes having height equal to $n+1$.

From these bijections we deduce some relations among the parameters of Motzkin and Schröder permutations.

We denote by PP_n the set of steep parallelogram polyominoes of half-perimeter n. The two operations that allow us to obtain all the polyominoes in PP_{n+1} from the ones in PP_n are the following:

1. add a cell onto the last column; therefore, $P' = \{(c_1, d_1), (c_2, d_2), \ldots, (c_m+1, d_m+1)\}$;
2. add a column of height $h \in [1, \ldots, c_m - 1]$ to the right of the last column; consequently:

$$P' = \{(c_1, d_1), (c_2, d_2), \ldots, (c_m, d'_m), (c_{m+1}, d_{m+1})\colon c_{m+1} = h,\ d'_m = d_m - h\}.$$

We denote $n(P)$ the width of P, $l(P)$ its height, $a(P)$ the height of its last column and, $f(P)$, the number of right-joined cells of P.

When we apply the above two operations, these parameters change as follows:

1. $n(P') = n(P),\ l(P') = l(P) + 1,\ a(P') = a(P) + 1,\ f(P') = f(P)$;
2. $n(P') = n(P) + 1,\ l(P') = l(P),\ a(P') = h,\ f(P') = f(P) + h$, for $h = 1, \ldots, a(P) - 1$.

The correspondence between these parameters and those of Motzkin permutations is now obvious: the half-perimeter is the length minus two, the height of the last column is the number of the active sites, the height is the number of right minima plus one, and the number of right-joined cells coincides with the inversion number.

Let us now denote the set of steep parallegram polyominoes having an n height by HP_n. The two operations that allow us to obtain all the polyominoes in HP_{n+1} from

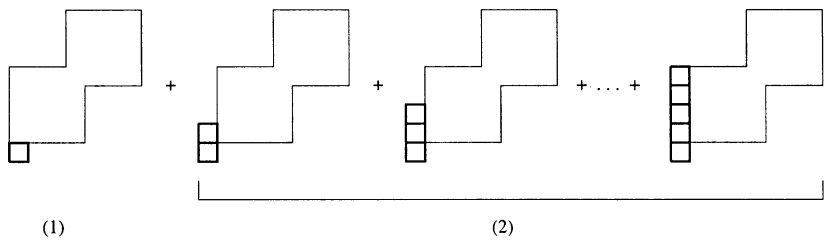

Fig. 5. The construction of steep parallelogram polyominoes according to the height.

the ones in HP_n are the following:

1. add a cell under the first column; therefore, $P'=\{(c_1+1,d_1+1),(c_2,d_2),\ldots,(c_m,d_m)\}$;
2. add a column of height $h\in[2,\ldots,c_1+1]$ to the left of the first column; consequently:

$$P'=\{(c'_1,d'_1),(c'_2,d'_2),\ldots,(c'_m,d'_m),(c'_{m+1},d'_{m+1})\},$$

where $c'_1=h$, $d'_1=1$, and $c'_i=c_{i-1}$, $d'_i=d_{i-1}$, for $i=2,\ldots,m+1$.

(see Fig. 5). We denote $a_1(P)$, the height of the first column of P. When we apply the above two operations, these parameters change as follows:

1. $n(P')=n(P)$, $l(P')=l(P)+1$, $a_1(P')=a_1(P)+1$, $f(P')=f(P)$;
2. $n(P')=n(P)+1$, $l(P')=l(P)+1$, $a_1(P')=h, f(P')=f(P)+h-1$, for $h=2,\ldots,a_1(P)$.

The correspondence between these parameters and those of Schröder permutations is now obvious: the height is the length, the height of the first column is the number of the active sites minus one, the number of right-joined cells coincides with the inversion number.

Consequently, we obtain the following relations among the parameters of Motzkin and Schröder permutations.

Proposition 6.4. *The number of $S(4231,4132)$ permutations, that is Schröder permutations, having length $n+1$, and k inversions is equal to the number of $S(321,3\bar{1}42)$ permutations, that is Motzkin permutations, having n right minima and k inversions.*

We wish to point out that this proposition extends the result of Proposition 4.6.

References

[1] E. Barcucci, A. Del Lungo, J.M. Fédou, R. Pinzani, Steep polyominoes, q-Motzkin numbers and q-Bessel functions, Discrete Math. 189 (1998) 21–42.

[2] M. Bousquet-Mélou, A method for the enumeration of various classes of column-convex polygons, Discrete Math. 154 (1996) 1–25.

[3] M. Delest, Polyominoes and animals: some recent results, J. Comput. Chem. 8 (1991) 3–18.

[4] M. Delest, J.M. Fédou, Enumeration of skew Ferrers diagrams, Discrete Math. 112 (1993) 65–79.

[5] S. Gire, Arbres, permutations à motifs exclus et cartes planaires: quelques problèmes algorithmiques et combinatoires, Thèse de l'Université de Bordeaux I, 1993.

[6] O. Guibert, Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young, Thèse de l'Université de Bordeaux I, 1995.
[7] D.E. Knuth, The Art of Computer Programming, 2nd Edition, Vol. 1, Addison-Wesley, Reading, MA, 1973.
[8] X.G. Viennot, A Survey of polyomino enumeration, in: P. Leroux, C. Reutenauer (Eds.), Proceedings Séries formelles et combinatoire algébrique, Publications du LACIM 11, Université du Québec, Montréal, 1992.
[9] J. West, Permutations with forbidden subsequences and stack-sortable permutations, Ph. D. Thesis, MIT, Cambridge, MA, 1990.
[10] J. West, Generating trees and the Catalan and Schröder numbers, Discrete Math. 146 (1995) 247–262.