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Steep polyominoes, q-Motzkin numbers and q-Bessel functions[1]

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Abstract

We introduce three definitions of q-analogs of Motzkin numbers and illustrate some combinatorial interpretations of these q-numbers. We relate the first class of q-numbers to the generating function for steep parallelogram polyominoes according to their width, perimeter and area. We show that this generating function is the quotient of two q-Bessel functions. The second class of q-Motzkin numbers counts the steep staircase polyominoes according to their area, while the third one enumerates the inversions of steep Dyck words. These enumerations allow us to illustrate various techniques of counting and q-counting. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

The Motzkin numbers M_n arise in a variety of combinatorial situations some of which are described in [17]. Furthermore, there are some nice algebraic relations between these numbers and some other quantities, such as Catalan numbers and trinomial coefficients [7].

Many q-generalizations of Catalan numbers have been defined and studied and many combinatorial interpretations of these q-numbers have been found. For instance, some of them are: the explicit q-Catalan numbers that enumerate Dyck words according to the major index [31,29,24]; Carlitz q-Catalan numbers, that count the inversions of Dyck words and Catalan permutations [21,28]; Polya–Gessel q-Catalan numbers that enumerate the parallelogram polyominoes according to their area [32,23] (see [21] for a survey).

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On the contrary, to the authors' knowledge, there are few definitions of q-analogs of Motzkin numbers. One of them is given in [8], and the combinatorial interpretation of this q-generalization is related to lattice paths.

In this paper, we introduce the following three q-generalizations of Motzkin numbers: $\bar{M}_n(x,q)$, $\tilde{M}_n(q)$, $M_n(q)$, which are similar to the definitions of Carlitz and Polya–Gessel q-Catalan numbers. We propose a combinatorial interpretation of each q-generalization. These interpretations also allow us to illustrate various recent techniques of counting and q-counting.

Section 2 of this paper contains the definitions of q-Motzkin numbers and q-Bessel functions, and the descriptions of some combinatorial objects. In Section 3, we use object grammars [18] to establish a relationship between q-Motzkin numbers $\bar{M}_n(x,q)$ and the generating function for steep parallelogram polyominoes according to their width, perimeter and area. By using a method based on the study of path pairs in the plane [19], we prove that the generating function is the quotient of two q-Bessel functions. In Section 4, we determine the generating function for steep staircase polyominoes according to their width and area and, by means of a recursive method illustrated in [11], we establish that the q-Motzkin numbers $\tilde{M}_n(q)$ enumerate these polyominoes according to their area. Finally, in Section 5, we enumerate steep Dyck words according to their length and inversion number by using a grammar that gives these words. We prove that the q-Motzkin numbers $M_n(q)$ count the inversions of the steep Dyck words. Moreover, we show that there is a relation between the generating function for these words according to their length, the descents number and the major index, and the q-Motzkin numbers $\bar{M}_n(q)$.

2. Notations and definitions

2.1. Polyominoes

Let us consider the plane $\pi = \mathbb{Z} \times \mathbb{Z}$. A *cell* is a unit square in π, a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to a translation. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite vertical strip of cells. The *area* of a polyomino is its number of cells, while its *height* and *width* are the numbers of its rows and columns, respectively.

Parallelogram polyominoes or *skew Ferrers diagrams* are defined by two non intersecting paths (except at their origin and extremity) having only north or east steps. For instance, Fig. 1(b) illustrates a parallelogram polyomino defined as having area 16, width 5 and height 6. Such polyominoes have been extensively studied. They are counted by Catalan numbers when they are enumerated according to their perimeter, and their generating function according to their area, width and height is related to q-Bessel functions and q-Catalan numbers. See [32,30,15] for their enumeration according to their perimeter and [27,14,9,10] with regard to their enumeration according to their area. More recently, Fédou and Rouillon [19] have developed a very simple

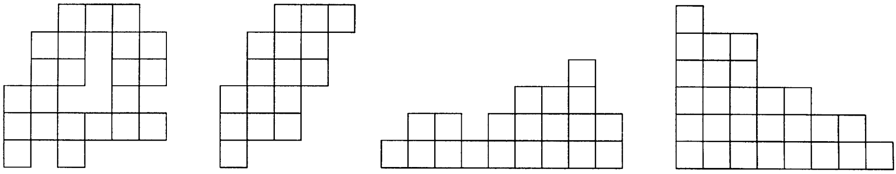

Fig. 1. A polyomino (a), a parallelogram polyomino (b), a staircase polyomino (c) and a Ferrers diagram.

explanation of their enumeration according to their area by means of q-Bessel functions.

A polyomino is *column-convex* if all its columns are connected. A *staircase* polyomino is a column-convex polyomino defined by a sequence of integers $(c_1, c_2, \ldots, c_k)$, such that

- for every $i \in [1..n]$, $c_{i+1} \leqslant c_i + 1$ and $c_1 = 1$,

where c_i is the number of cells of the ith column (see Fig. 1(c)). They are counted by Catalan numbers according to their width, while their generating function according to their area and width is related to q-Catalan numbers. See [13] for their enumeration according to their width, and [10] for their enumeration according to their area and width.

A *Ferrers diagram* is a column-convex polyomino defined by a decreasing sequence of integers $(c_1, c_2, \ldots, c_k)$, where c_i is the number of cells of the ith column (see Fig. 1(d)). These polyominoes are related to the well-known partitions of integers, and have been extensively studied (see, for instance, [2]).

2.1.1. Steep parallelogram and staircase polyominoes

Let us call *south border* and *north border* the two paths defining a parallelogram polyomino. A (*lower*) *steep parallelogram polyomino* is a parallelogram polyomino whose south border has no pairs of subsequent horizontal steps (see Fig. 2(a)). We denote the set of steep parallelogram polyominoes by SP. Analogously, an *upper steep parallelogram polyomino* has no pairs of subsequent horizontal steps (see Fig. 2(b)) on its north border.

A *steep staircase polyomino* is a staircase polyomino having any two adjacent columns of different lengths (i.e., for any $i \in [1..n]$, $c_{i+1} < c_i$ or $c_{i+1} = c_i + 1$, see Fig. 2(c)). We denote the set of steep staircase polyominoes by SS.

2.2. Words

Let X be an alphabet. We define the free monoid generated by X, denoted X^*, as the set of finite words written with the letters taken from X. The product of $u = u_1 \ldots u_p \in X^*$ and $v = v_1 \ldots v_q \in X^*$ is defined as the *concatenation* of these words: $uv = u_1 \ldots u_p v_1 \ldots v_q$. The word u is called a *left factor* of the word $w = uv$. The *empty word* is denoted by ε. The number of times the letter $a \in X$ occurs in the word w is denoted by $|w|_a$, while the length of w is denoted by $|w| = \sum_{a \in X} |w|_a$.

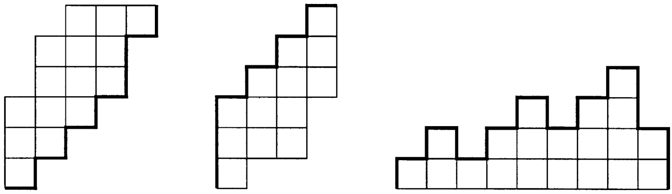

Fig. 2. A (lower) steep (a) an upper steep parallelogram polyomino (b), a steep staircase polyomino (c).

The set of *Dyck words* is the set of words w on $X^* = \{x, \bar{x}\}^*$ characterized by the following two conditions:

- for any left factor u of w, $|u|_x \geqslant |u|_{\bar{x}}$,
- $|w|_x = |w|_{\bar{x}}$.

The set of *Motzkin words* is the set of words w on $X^* = \{x, a, \bar{x}\}^*$ characterized by the previous two conditions.

Let $w = w_1 w_2 \dots w_n \in$ be a Dyck word or a Motzkin word. We assume that $x < a < \bar{x}$. An *inversion* of w is a $w_i w_j$ pair, where $w_i > w_j$, and $i < j$. The *down set* of w is the following:

$$D(w) = \{i\colon\ w_i > w_{i+1}, 1 \leqslant i \leqslant n-1\}.$$

Two classical parameters related to the down set are the *major index* and the *number of descents*:

$$\mathrm{maj}(w) = \sum\{i\colon\ i \in D(w)\}, \qquad \mathrm{des}(w) = |D(w)|.$$

The number of Dyck words of length $2n$ is equal to the nth Catalan number; the enumeration of these words according to their inversion number and down set parameters is related to q-Catalan numbers [21].

The number of Motzkin words of length n is equal to the nth Motzkin number [17]. In [8], it is shown that the distribution of the major index over the Motzkin words gives a nice q-analog of Motzkin numbers.

A *path* is a sequence of points in $\mathbb{N} \times \mathbb{N}$. A *step* of a path is a pair of two consecutive points in the path. Each Dyck word $w = w_1 \dots w_{2n}$ codifies a path $w = (s_0, s_1, \dots, s_{2n})$ only having northeast $(s_i = (x, y), s_{i+1} = (x+1, y+1))$ or southeast $(s_i = (x, y), s_{i+1} = (x+1, y-1))$ steps. Each northeast (southeast) step (s_{i-1}, s_i) corresponds to the letter $w_i = x$ $(w_i = \bar{x})$. These paths are called *Dyck paths*. Each Motzkin word codifies a path having northeast, southeast and east $(s_i = (x, y), s_{i+1} = (x+1, y))$ steps. The east steps correspond to the letters $w_i = a$. These paths are called *Motzkin paths*.

From a geometrical point of view, the inversion number of a Dyck word w is equivalent to the area of the Ferrers diagram which lies between the Dyck path corresponding to w and the path corresponding to $x \dots x\bar{x} \dots \bar{x}$ (see Fig. 3).

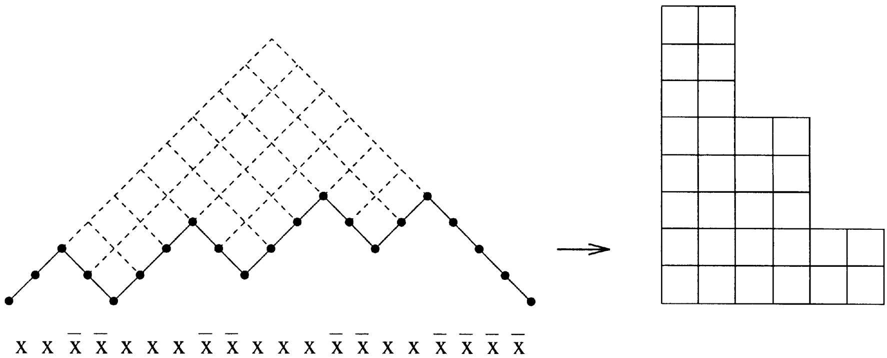

Fig. 3. A Dyck word and its corresponding Dyck path and Ferrers diagram.

2.2.1. *Steep Dyck words*

The set of *steep Dyck words* is the set of Dyck words w such that

- $w \neq \varepsilon$ and w does not contain any $x\bar{x}x$ factor.

We denote the set of the steep Dyck words by SW. The paths codified by a steep Dyck word have no 'isolated' southeast steps (i.e., if (s_{i-1}, s_i) is a southeast step, then (s_{i-2}, s_{i-1}) or (s_i, s_{i+1}) are also southeast steps). For instance, the Dyck path in Fig. 3 is a steep Dyck path. Moreover, the column lengths in the Ferrers diagrams that correspond to steep Dyck words are such that: for any $i \in [1..n]$, $c_{i+1} = c_i$ or $(c_{i+1} \leqslant c_i - 1$ and $c_i = c_{i-1})$.

2.3. *q-calculus*

The following are the q-calculus notations most commonly used:

- $[n] = (1 - q^n)/(1 - q)$,
- $(a;q)_n$ denotes the $\prod_{i=1}^{n}(1 - aq^{i-1})$ product and $(a;q)_0 = 1$,
- $(q)_n = \prod_{i=1}^{n}(1 - q^i)$ is a simplified notation of $(q;q)_n$,
- $\left[{n \atop k}\right] = (q)_n/((q)_k(q)_{n-k})$,
- $(yq)_n = \prod_{i=1}^{n}(1 - yq^i)$ is a simplified notation of $(yq;q)_n$.

We also need $(1/q)$-analog and so we denote

- $(1/q)_n = \prod_{i=1}^{n}(1 - 1/q^i)$.

2.3.1. *q-Motzkin numbers*

The Motzkin numbers M_n are traditionally defined by the following recurrence relation:

$$M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-k-1}, \qquad M_0 = 1.$$

Therefore, the generating function for Motzkin numbers $M(x)=\sum_{n\geqslant 0} M_n x^n$ satisfies $M(x)=1+xM(x)+x^2M^2(x)$ and so

$$M(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

Motzkin numbers can be expressed in terms of the Catalan numbers $C_n=(1/(n+1))\binom{2n}{n}$ by

$$M_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k,$$

and vice versa

$$C_{n+1}=\sum_{k\geqslant 0}\binom{n}{k}M_k$$

(see for example [7]). Several q-Catalan numbers have been defined and studied:

- Carlitz q-Catalan numbers, proposed by Carlitz and Riordan [12] and studied in detail by many authors [1,21–23,33]:

$$C_{n+1}(q)=\sum_{k=0}^{n}C_k(q)C_{n-k}(q)q^{(k+1)(n-k)}, \qquad C_0(q)=1$$

$$\tilde{C}_{n+1}(q)=\sum_{k=0}^{n}\tilde{C}_k(q)\tilde{C}_{n-k}(q)q^{k}, \qquad \tilde{C}_0(q)=1$$

  They count the inversions of Dyck words and Catalan permutations [21,28], respectively, and

$$\tilde{C}_n(q)=q^{\binom{n}{2}}C_n(q^{-1}).$$

- The explicit q-Catalan numbers

$$\frac{1}{[n+1]}\begin{bmatrix}2n\\ n\end{bmatrix}$$

  introduced by MacMahon [31] have been studied by Krattenthaler [29], Gessel and Stanton [24]. They enumerate Dyck words according to the major index. A generalization of these q-analogs is given by Fürlinger and Hofbauer [21]. They enumerate Dyck words according to the number of descents and a refinement of the major index.
- The q-Catalan numbers introduced by Polya [32] and Gessel [23]

$$P_{n+1}(x,q)=xqP_n(x,q)+P_n(xq,q)+\sum_{k=0}^{n-1}P_k(xq,q)P_{n-k}(x,q),$$

  where $P_0(x,q)=P_1(x,q)=0$ and $P_2(x,q)=xq$, count the parallelogram polyominoes having perimeter $2n$ according to their area.

A q-analog of Motzkin numbers is defined by Bonin et al. [8]. The authors prove that the generating function $MS_{n,k}(q)$ for the Motzkin words having length n and k letters x according to the major index (with $x<a<\bar{x}$) is

$$MS_{n,k}(q)=\frac{1}{k+1}\begin{bmatrix}2k\\k\end{bmatrix}\begin{bmatrix}n\\2k\end{bmatrix},$$

and so they obtain the following nice q-analog:

$$MS_n(q)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{n+1}\begin{bmatrix}2k\\k\end{bmatrix}\begin{bmatrix}n\\2k\end{bmatrix}.$$

We now introduce three definitions of q-Motzkin numbers different from the previous one and, in the following sections, we go on to determine some combinatorial interpretations of these q-numbers.

Definition 2.1. We define the q-analogs of Motzkin numbers as

$$M_{n+1}(q)=M_n(q)+\sum_{k=0}^{n-1}M_k(q)M_{n-k-1}(q)q^{(k+2)(n-k)},\qquad M_0(q)=1, \tag{2.1}$$

$$\tilde{M}_{n+1}(q)=\tilde{M}_n(q)q^{n+2}+\sum_{k=0}^{n-1}\tilde{M}_k(q)\tilde{M}_{n-k-1}(q)q^{k+2},\qquad \tilde{M}_0(q)=q, \tag{2.2}$$

$$\bar{M}_{n+1}(x,q)=\bar{M}_n(xq,q)+\sum_{k=0}^{n-1}\bar{M}_k(x,q)\bar{M}_{n-k-1}(xq,q),\qquad \bar{M}_0(x,q)=xq. \tag{2.3}$$

We wish to point out that the q-Motzkin number of Bonin et al. are similar to MacMahon q-Catalan numbers, while our first and second definitions are similar to Carlitz q-Catalan numbers, and the third one corresponds to the Polya–Gessel q-Catalan numbers. Moreover,

$$\tilde{M}_n(q)=q^{\binom{n+2}{2}}M_n(q^{-1}),$$

and, by putting $q=1$, $x=1$, we obtain the Motzkin numbers $(M_n(1)=\tilde{M}_n(1)=\bar{M}(1,1)=M_n)$.

2.3.2. q-Bessel functions

The Bessel functions of order v, with v integer, are commonly defined by

$$\mathscr{J}_v(x)=\sum_{n\geqslant 0}\frac{(-1)^n(x/2)^{2n+v}}{n!(n+v)!}.$$

Several q-analogs of these functions were first defined and studied by Jackson [26] and subsequently by Ismail [25], and Gessel and Stanton [24]. They are all defined as

$$J_v(x;q)=\sum_{n\geqslant 0}(-1)^n a_{n,v}\frac{x^{n+v}}{(q)_n(q)_{n+v}}.$$

For instance, the q-Bessel functions that count parallelogram polyominoes are

$$J_0(x,y;q)=\sum_{n\geqslant 0}\frac{(-1)^n q^{\binom{n+1}{2}}x^n}{(q)_n(yq)_n},$$

$$J_1(x,y;q)=\sum_{n\geqslant 1}\frac{(-1)^{n-1} yq^{\binom{n+1}{2}}x^n}{(q)_{n-1}(yq)_n}.$$

More precisely, the generating function for parallelogram polyominoes according to their area, width and height is $J_1(x,y;q)/J_0(x,y;q)$ (see [14,10]). This function is related to Polya–Gessel q-Catalan numbers [23,32].

Definition 2.2. We define the q-analogs of the Bessel functions $\bar{J}_0$ and $\bar{J}_1$ as

$$\bar{J}_0(x,y;q)=\sum_{n\geqslant 0}\frac{(-1)^n x^n}{(q)_n(yq)_n},$$

$$\bar{J}_1(x,y;q)=\sum_{n\geqslant 1}\frac{yx^n}{(q)_{n-1}(yq)_n}.$$

The amazing result is that these q-Bessel functions arise when counting steep parallelogram polyominoes and are related to some q-Motzkin numbers.

We also need the $(q;1/q)$-Bessel functions and so we define:

Definition 2.3.

$$\tilde{J}_0(x,y;q,1/q)=\sum_{n\geqslant 0}\frac{x^n q^{\binom{n+1}{2}}}{(qy)_n(1/q)_n},$$

$$\tilde{J}_1(x,y;q,1/q)=\sum_{n\geqslant 1}\frac{x^n yq^{\binom{n}{2}}}{(qy)_n(1/q)_{n-1}}.$$

We wish to point out that these two functions belong to a strange ring of formal power series in x and y, with coefficient in the Laurent series in $1/q$, defined as

$$\sum_{m,n\geqslant 0}\left(\sum_{k\leqslant k_0} q^k\right) x^m y^n,$$

in which the following identities simultaneously hold:

$$\frac{1}{1-qy}=\sum_{n\geqslant 0} q^n y^n,$$

$$\frac{1}{1-1/q}=\sum_{n\geqslant 0} 1/q^n.$$

3. Steep parallelogram polyominoes, *q*-Bessel functions and *q*-Motzkin numbers

In this section, we enumerate steep parallelogram polyominoes according to their width, perimeter and area and we show that their generating function is related both to q-Motzkin numbers $\bar{M}_n(x,q)$ and q-Bessel functions $\bar{J}_0, \bar{J}_1$.

3.1. Results

A first and simple result is the enumeration of steep parallelogram polyominoes according to their perimeter.

Theorem 3.1. *Steep parallelogram polyominoes are counted by Motzkin numbers according to their perimeter.*

This is easily shown by means of the classical bijection between parallelogram polyominoes and Dyck words (see [16]). Object grammars [18] (see next section) also provide a very satisfactory explanation of this. Moreover, this allows us to deduce a functional equation satisfied by the generating function $F_{SP}(x,t,q)$ of steep parallelogram polyominoes according to their width (x) perimeter (t) and area (q).

Proposition 3.2.

$$F_{SP}(x,t,q) = xt^2q + tF_{SP}(xq,t,q) + F_{SP}(x,t,q)F_{SP}(xq,t,q). \tag{3.1}$$

From this functional equation, the relationship between $F_{SP}(x,t,q)$ and the q-Motzkin numbers $\bar{M}_n(x,q)$ follows:

Theorem 3.3.

$$F_{SP}(x,t,q) = t^2 \sum_{n\geqslant 0} \bar{M}_n(x,q)t^n. \tag{3.2}$$

The relationship between the area generating function of steep parallelogram polyominoes and q-Bessel functions is not very obvious. Let $G_{SP}(x,t,q)$ be the generating function for steep parallelogram polyominoes according to their width (x) height (y) and area (q).

Theorem 3.4. *The generating function for steep parallelogram polyominoes according to their width, height and area is*

$$G_{SP}(x,y,q) = \frac{\bar{J}_1(x,y,q)}{\bar{J}_0(x,y,q)},$$

and so we get:

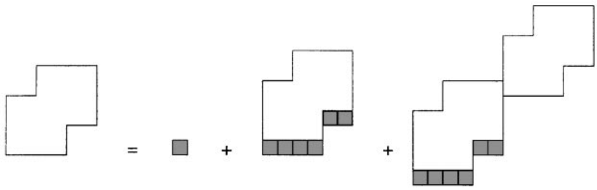

Fig. 4. An object grammar for steep parallelogram polyominoes.

Corollary 3.5. *The generating function for steep parallelogram polyominoes according to their width, perimeter and area is*

$$F_{\mathrm{SP}}(x,t,q)=\frac{\bar{J}_1(tx,t,q)}{\bar{J}_0(tx,t,q)}.$$

Thus, this result is very similar to the generating function for ordinary parallelogram polyominoes, but it involves some other q-Bessel functions. It can be deduced from the functional q-equation (3.1). We now give a combinatorial proof based on the study of path pairs in the plane as proposed by Fédou and Rouillon [19].

3.2. Proofs

Steep parallelograms polyominoes can be recursively defined by the two basic operations illustrated in Fig. 4.

- The former operation has arity 1 and consists in adding a cell to each column of the polyomino.
- The latter has arity 2 and consists in gluing two polyominoes together along only one cell, and adding a cell to each column of the first polyomino.

It is clear that steep parallelogram polyominoes are generated unambiguously by these operations and the unit cell. This is an example of what Dutour and Fédou call *object grammar* [18]. From the object grammar shown in Fig. 4, Eq. (3.1) can be easily deduced. By replacing q and x by 1, we obtain the following equation for the generating function for steep parallelogram polyominoes according to their perimeter:

$$F_{\mathrm{SP}}(1,t,1)=t^2+tF_{\mathrm{SP}}(1,t,1)+F_{\mathrm{SP}}^2(1,t,1),$$

and it is therefore obvious that such polyominoes are counted by Motzkin numbers according to their perimeter. It is worth noting that the object grammar illustrated in Fig. 4 induces natural bijections among steep polyominoes and Motzkin words (or 1–2 trees), and all kinds of objects generated by a unary operation, a binary operation and one basic object (see Fig. 5 for some isomorphic object grammars).

If we denote $F_{\mathrm{SP}}(x,t,q)=\sum_{n\geqslant 0}F_n(x,q)t^n$, then we obtain

$$F_n(x,q)=F_{n-1}(xq,q)+\sum_{k=2}^{n-2}F_k(xq,q)F_{n-k}(x,q),$$

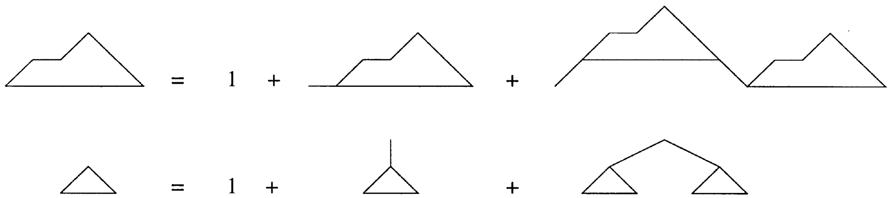

Fig. 5. Isomorphic object grammars for Motzkin path and 1–2 trees.

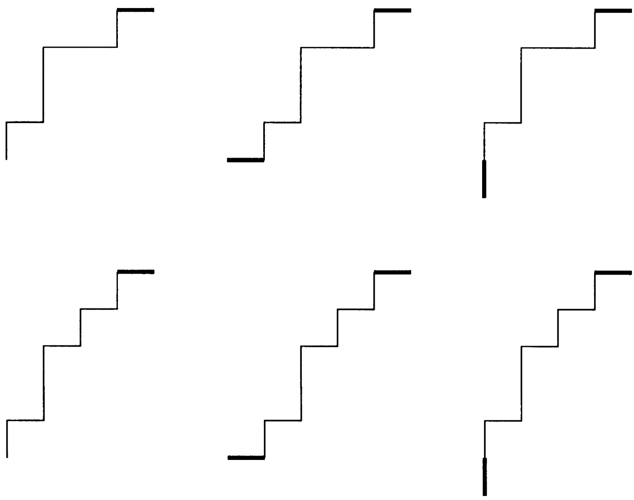

Fig. 6. Different kinds of paths from $\mathscr{C}_4$, ${}_{-}\mathscr{C}_5$, ${}_{|}\mathscr{C}_4$, $\tilde{\mathscr{C}}_4$, ${}_{-}\tilde{\mathscr{C}}_5$, ${}_{|}\tilde{\mathscr{C}}_4$.

where $F_0(x,q)=F_1(x,q)=0$ and $F_2(x,q)=xq$. From the definition of q-Motzkin numbers, it follows that $F_n(x,q)=\bar{M}_{n-2}(x,q)$, and we obtain (3.2).

The proof of Theorem 3.4 is based on the study of some path pairs in the plane according to [19]. Let $\mathscr{C}$ be the set of the plane paths that are either empty or made up of a sequence of north and east unit steps, and that begin at the origin and end with an east step. Let $\tilde{\mathscr{C}}$ be the set of these paths having no pairs of consecutive east steps (see Fig. 6). $\mathscr{C}_n$ and $\tilde{\mathscr{C}}_n$ denote the paths of $\mathscr{C}$ and $\tilde{\mathscr{C}}$ having width n. We obtain some useful results by means of a simple proposition:

Proposition 3.6. *The generating functions for the paths of $\mathscr{C}_n$ and $\tilde{\mathscr{C}}_n$ according to their area, height and width are*

$$\mathrm{SG}_{\mathscr{C}_n}(x,y,q) = \frac{x^n}{(yq)_n}, \tag{3.3}$$

$$\mathrm{SG}_{\tilde{\mathscr{C}}_n}(x,y,q) = \frac{q^{\binom{n}{2}} x^n y^{n-1}}{(yq)_n}, \tag{3.4}$$

respectively.

Furthermore, we consider

- $_{-}\mathscr{C}$ the set of the paths of $\mathscr{C}$ beginning with an east step.
- $_{|}\mathscr{C}$ the set of the paths of $\mathscr{C}$ beginning with a north step.
- $_{-}\tilde{\mathscr{C}}$ the set of the paths of $\tilde{\mathscr{C}}$ beginning with an east step.
- $_{|}\tilde{\mathscr{C}}$ the set of the paths of $\tilde{\mathscr{C}}$ beginning with a north step.

We index these sets by n when we only take paths of width n into consideration. We then obtain the following generating functions.

Corollary 3.7. *The generating functions for the paths of* $_{-}\mathscr{C}_n$, $_{|}\mathscr{C}_n$, $_{-}\tilde{\mathscr{C}}_n$ *and* $_{|}\tilde{\mathscr{C}}_n$ *according to their area, height and width are*

$$\mathrm{SG}_{_{-}\mathscr{C}_n}(x,y,q) = \frac{x^n}{(yq)_{n-1}},$$

$$\mathrm{SG}_{_{|}\mathscr{C}_n}(x,y,q) = \frac{q^n\, y\, x^n}{(yq)_n},$$

$$\mathrm{SG}_{_{-}\tilde{\mathscr{C}}_n}(x,y,q) = \frac{q^{\binom{n}{2}}\, y^{n-1}\, x^n}{(yq)_{n-1}},$$

$$\mathrm{SG}_{_{|}\tilde{\mathscr{C}}_n}(x,y,q) = \frac{q^{\binom{n+1}{2}}\, y^n\, x^n}{(yq)_n},$$

respectively.

We now examine path pairs, and define:

Definition 3.1.

$$\mathscr{D} = \bigcup_{n\geqslant 0} \mathscr{C}_n \times \tilde{\mathscr{C}}_n,$$

$$\bar{\mathscr{D}} = \bigcup_{n\geqslant 1} {}_{|}\mathscr{C}_n \times {}_{-}\tilde{\mathscr{C}}_n.$$

Some examples of these path pairs are given in Fig. 7.

In order to prove Theorem 3.4, we count these sets according to their width, height and positive area as far as the former path is concerned (the generating functions are given in Corollary 3.7), while we count the latter negatively according to its area. We only need to replace x by 1, y by 1 and q by $1/q$ in Corollary 3.7.

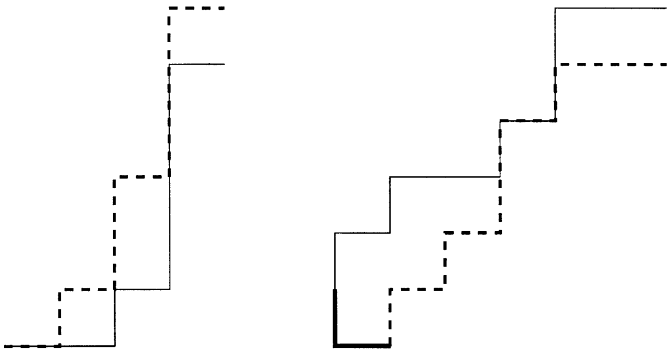

Fig. 7. A pair of paths belonging to $\mathscr{D}$ and $\bar{\mathscr{D}}$, respectively.

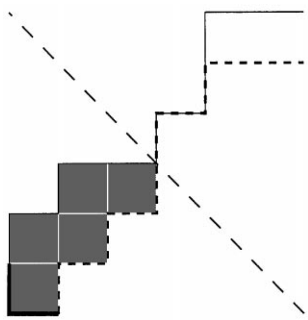

Fig. 8. Factorization of a ($\bar{\mathscr{D}}$)-path into a parallelogram polyomino and a ($\mathscr{D}$)-path.

We deduce the following lemma by relating the functions $\tilde{J}_0(x,y;q,1/q)$ and $\tilde{J}_1(x,y;q,1/q)$ to $\mathscr{D}$ and $\bar{\mathscr{D}}$.

Lemma 3.8. *The generating functions of the sets $\mathscr{D}$ and $\bar{\mathscr{D}}$ according to x, y, q and $1/q$ are given by*

$$G_{\mathscr{D}} = \tilde{J}_0(x,y;q,1/q)$$

and

$$G_{\bar{\mathscr{D}}} = \tilde{J}_1(x,y;q,1/q).$$

Since each element in $\bar{\mathscr{D}}$ can be factorized in a unique way into a steep parallelogram polyomino and into an element of $\mathscr{D}$ (see Fig. 8), the equality $\bar{J}_1(x,y;q) = G_{SP} \times \bar{J}_0(x,y;q)$ follows. From this, we deduce Theorem 3.4 by replacing each $1/(1-1/q^i)$ by $-q^i/(1-q^i)$. This transformation is possible only because the coefficient of $x^i y^j$ is a polynomial in q, and this gives the same development in $q=\infty$ and $q=0$.

4. Steep staircase polyominoes and q-Motzkin numbers

In this section, we determine the generating function for steep staircase polyominoes according to their area and width and show that it is related to the q-Motzkin numbers $\tilde{M}_n(q)$.

4.1. Results

Let $G_{SS}(s,x,q)$ be the generating function for steep staircase polyominoes according to the area (q), width (x) and length of the rightmost column (s). We often denote the function $G_{SS}(s,x,q)$ by $G_{SS}(s)$ for brevity's sake. By using the 'Temperley methodology' [6,11,34] we deduce:

Proposition 4.1. *The generating function $G_{SS}(s,x,q)$ verifies the following functional equation*:

$$G_{SS}(s) = sxq + \frac{sxq}{1-sq} G_{SS}(1) + x\left(sq - \frac{1}{1-sq}\right) G_{SS}(sq). \tag{4.1}$$

By solving this functional equation we obtain the generating function.

Theorem 4.2. *The generating function $G_{SS}(s,x,q)$ is given by*

$$G_{SS}(s) = \frac{E_1(s)E_0(1) - E_1(1)E_0(s) + E_1(1)}{E_0(1)},$$

with

$$E_0(s) = 1 - sxq \sum_{n\geqslant 0} (-1)^n \frac{x^n q^n}{(sq)_{n+1}} \prod_{k=0}^{n-1} (1 - sq^{k+1} - s^2 q^{2(k+1)}),$$

and

$$E_1(s) = sxq \sum_{n\geqslant 0} (-1)^n \frac{x^n q^n}{(sq)_n} \prod_{k=0}^{n-1} (1 - sq^{k+1} - s^2 q^{2(k+1)}).$$

From this theorem it follows that:

Theorem 4.3. *The generating function for steep staircase polyominoes having width n according to their area is equal to q-Motzkin number $\tilde{M}_{n-1}(q)$, i.e.*

$$G_{SS}(1,x,q) = x \sum_{n\geqslant 0} \tilde{M}_n(q) x^n. \tag{4.2}$$

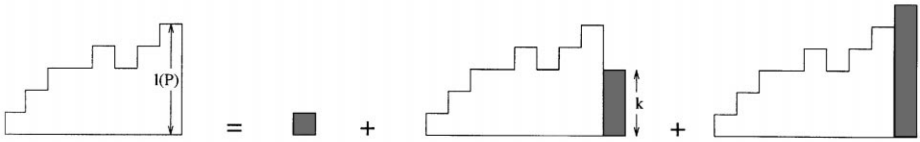

Fig. 9. A recursive description of steep staircase polyominoes.

Moreover,

$$G_{\mathrm{SS}}(1,x,q)=\frac{E_1(1,x,q)}{E_0(1,x,q)}=\cfrac{1+xq}{1+xq-\cfrac{xq(1+xq^2)}{1+xq^2-\cfrac{xq^2(1+xq^3)}{1+xq^3-\cfrac{xq^3(1+xq^4)}{\cdots}}}}-1. \quad (4.3)$$

Corollary 4.4. *The number of steep staircase polyominoes having width n is equal to $(n-1)$th Motzkin number.*

4.2. Proof

Let P be a steep staircase polyomino. We denote the length of its rightmost column by $l(P)$, its width $h(P)$ and its area $a(P)$. The generating function for steep staircase polyominoes according to the above-listed parameters is

$$G_{\mathrm{SS}}(s,x,q)=\sum_{P\in \mathrm{SS}} s^{l(P)}x^{h(P)}q^{a(P)}.$$

Fig. 9 illustrates a recursive description of steep staircase polyominoes.

A steep staircase polyomino P can be recursively defined in the following way:
1. P is made up of only one cell;
2. P is obtained by gluing a steep staircase polyomino P' to a column C of length $k\leqslant l(P')-1$ (therefore: $a(P)=a(P')+k$, $l(P)=k$, $h(P)=h(P')+1$);
3. P is obtained by gluing a steep staircase polyomino P' to a column C of length $l(P')+1$ (therefore $a(P)=a(P')+l(P')+1$, $l(P)=l(P')+1$, $h(P)=h(P')+1$).

We refer these constructions to the generating function $G_{\mathrm{SS}}(s,x,q)$, and obtain:

$$G_{\mathrm{SS}}(s,x,q)=sxq+\sum_{P\in \mathrm{SS}}\sum_{k=1}^{l(P)-1} s^k x^{h(P)+1}q^{a(P)+k}+\sum_{P\in \mathrm{SS}} s^{l(P)+1}x^{h(P)+1}q^{i(P)+l(P)+1},$$

from which we deduce functional equation (4.1).

Let us now solve functional equation (4.1) by means of the following lemma [11]:

Lemma 4.5. *Let $\mathscr{R}=\mathbb{R}[[s,x,q]]$ be the algebra of the formal power series in variables s,x and q with real coefficients, and let $\mathscr{A}$ be a sub-algebra of $\mathscr{R}$ such that the series converge for $s=1$.*

Let $X(s,x,q)$ be a formal power series in $\mathscr{A}$. We assume that

$$X(s)=xe(s)+xf(s)X(1)+xg(s)X(sq),$$

where $e(s), f(s)$ *and* $g(s)$ *are some given power series in* $\mathscr{A}$. *Then*

$$X(s)=\frac{E(s)+E(1)F(s)-E(s)F(1)}{1-F(1)},$$

where

$$E(s)=\sum_{n\geqslant 0} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})e(sq^n),$$

and

$$F(s)=\sum_{n\geqslant 0} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})f(sq^n).$$

By means of Lemma 4.1 and Eq. (4.1), we get the generating function $G_{SS}(s,x,q)$ of Theorem 4.2.

Let us now take the generating function $G_{SS}(1,x,q)$ into consideration. The proof of Theorem 4.3 is based on the study of the functions $E_1(1,x,q)$ and $E_0(1,x,q)$. By means of some computations, we obtain

Lemma 4.6. *The functions* $E_0(x,y,q)$ *and* $E_1(x,y,q)$ *satisfy the following equations*:

$$E_0(1,xq,q)=E_0(1,x,q)+E_1(1,x,q),$$

$$xq\,E_1(1,xq,q)=E_1(1,x,q)-xq\,E_0(1,x,q).$$

From Lemma 4.6, we deduce that

$$G_{SS}(1,xq,q)=\frac{xq\,E_1(1,xq,q)}{xq\,E_0(1,xq,q)}=\frac{E_1(1,x,q)-xq\,E_0(1,x,q)}{E_0(1,x,q)+E_1(1,x,q)},$$

and so we obtain

$$G_{SS}(1,x,q)=xq+xq\,G_{SS}(1,xq,q)+xq\,G_{SS}(1,x,q)G_{SS}(1,xq,q). \tag{4.4}$$

If we denote $G_{SS}(1,x,q)=\sum_{n\geqslant 0} S_n(q)\,x^n$, then we get

$$S_n(q)=S_{n-1}(q)q^n+\sum_{k=1}^{n-2} S_k(q)S_{n-k-1}(q)q^{k+1}, \qquad S_1(q)=q.$$

From the q-Motzkin numbers' definition, it follows that $S_n(q)=\tilde{M}_{n-1}(q)$ and we obtain (4.2). If $A(1,x,q)=1+G_{SS}(1,x,q)$, then

$$A(1,x,q)=\frac{1+xq}{1+xq-xqA(1,xq,q)},$$

so the continued fraction (4.3) follows.

By putting $q=1$ into Eq. (4.4), we deduce

$$G_{SS}(1,x,1)=x+x\,G_{SS}(1,x,1)+x\,G_{SS}^2(1,x,1).$$

Hence, the number $S_n(1)$ of steep staircase polyominoes having width n is equal to the $(n-1)$th Motzkin number.

5. Steep Dyck words and q-Motzkin numbers

In this section, we enumerate steep Dyck words according to their length and inversion number and we show that their generating function is related to the q-Motzkin numbers $M_n(q)$.

5.1. Results

Theorem 5.1. *The number of steep Dyck words having length $2n+2$ is equal the nth Motzkin number. Moreover, the number of steep Dyck words having length $2n+2$ and k descents is $1/(k+1)\binom{2k}{k}\binom{n}{2k}$.*

Let $G_{\mathrm{SW}}(x,q)$ be the generating function for steep Dyck words according to their inversion number (q) and length (x).

Theorem 5.2. *The generating function for steep Dyck words having length $2n$ according to their inversion number is equal to the q-Motzkin number $M_{n-1}(q)$, i.e.*

$$G_{\mathrm{SW}}(1,x,q)=x\sum_{n\geqslant 0} M_n(q)\,x^n. \tag{5.1}$$

In [21], the following refinement of the parameter major index over the Dyck words is studied. Let $w=w_1w_2\ldots w_{2n}$ be a Dyck word,

$$\alpha(w)=\sum_{i\in D(w)} |\{j\leqslant i\colon\ w_j=x\}|, \qquad \beta(w)=\sum_{i\in D(w)} |\{j\leqslant i\colon\ w_j=\bar{x}\}|.$$

where $D(w)$ is the down set of w. Obviously $\alpha(w)+\beta(w)=maj(w)$. Fürlinger and Hofbauer show that there is the following nice relation between the 3-variate q-Catalan numbers

$$C_n(s,a,b)=\sum_{w\in D_n} s^{des(w)}a^{\alpha(w)}b^{\beta(w)},$$

(where D_n is the set of Dyck words of length n) and Polya–Gessel q-Catalan numbers $P_n(s,q)$:

$$P_{n+1}(s,q)=sq^nC_n(s,q,q^{-1}).$$

If we consider the same q-analogs on the steep Dyck words

$$\hat{M}_n(s,a,b)=\sum_{w\in SW_n} s^{des(w)}a^{\alpha(w)}b^{\beta(w)},$$

we obtain that:

Theorem 5.3. *The q-analog $\hat{M}_n(s,a,b)$ is such that*

$$\hat{M}_{n+1}(s,a,b)=\hat{M}_n(sa,a,b)+s\sum_{k=0}^{n-1}(ab)^{k+1}\hat{M}_k(sa,a,b)\hat{M}_{n-k}(s(ab)^{k+1},a,b), \tag{5.2}$$

where $\hat{M}_1(s,a,b)=1$. *Moreover,*

$$\bar{M}(s,q)=sq^{n+1}\,\hat{M}_{n+1}(s,q,q^{-1}). \tag{5.3}$$

We wish to point out that, if $F_n(s,q)$ is the generating function for steep parallelogram polyominoes of perimeter $2n$ according to their width and area, then from Theorem 3.3 it follows that

$$F_{n+1}(s,q)=sq^n\,\hat{M}_n(s,q,q^{-1}).$$

5.2. Proof

The non-empty Dyck words are generated by the following grammar:

$$D' \to x\bar{x} \mid x\bar{x}\,D' \mid x\,D'\,\bar{x} \mid x\,D'\,\bar{x}\,D'.$$

Steep Dyck words are different from ε and do not contain any $x\bar{x}x$ sequence. Since this sequence is generated by the rule $D' \to x\bar{x}D'$, the steep Dyck words are generated by the algebraic grammar:

$$S \to x\bar{x} \mid x\,S\,\bar{x} \mid x\,S\,\bar{x}\,S. \tag{5.4}$$

The Motzkin words are generated by the following algebraic grammar:

$$M \to \varepsilon \mid a\,M \mid x\,M\,\bar{x}\,M.$$

By comparing this grammar with steep Dyck words' grammar we deduce that the steep Dyck words having length $2n+2$ are in bijection with the Motzkin words having length n. Therefore, the number of steep Dyck words having length $2n+2$ is equal to nth Motzkin number. Moreover, the letters $\bar{x}$ of the Motzkin words correspond to the descents of the steep Dyck words and so the number of steep Dyck words having length $2n+2$ and k descents is $1/(k+1)\binom{2k}{k}\binom{n}{2k}$.

Let SW_n be the set of the steep Dyck words having length $2n$. From grammar (5.4) it follows that a word $w\in\mathrm{SW}_{n+1}$ can be decomposed into $w=xw_1\bar{x}$ with $w_1\in\mathrm{SW}_n$, or $w=xw_1\bar{x}w_2$ with $w_1\in\mathrm{SW}_k$ and $w_2\in\mathrm{SW}_{n-k}$ for some k with $1\leqslant k\leqslant n-1$. In the first case we have:

$$\mathrm{inv}(w)=\mathrm{inv}(w_1),$$

$$\mathrm{des}(w)=\mathrm{des}(w_1),$$

$$\alpha(w)=\alpha(w_1)+\mathrm{des}(w_1),$$

$$\beta(w)=\beta(w_1),$$

and in the second one:

$$\text{inv}(w) = \text{inv}(w_1) + (k+1)(n-k) + \text{inv}(w_2),$$

$$\text{des}(w) = \text{des}(w_1) + 1 + \text{des}(w_2),$$

$$\alpha(w) = \alpha(w_1) + \text{des}(w_1) + (k+1) + \alpha(w_2) + (k+1)\,\text{des}(w_2),$$

$$\beta(w) = \beta(w_1) + (k+1) + \beta(w_2) + (k+1)\text{des}(w_2).$$

Let $G_{\text{SW}}(x,q) = \sum_{n\geqslant 0} G_n(q)\,x^n$ be the generating function for steep Dyck words according to their length and inversion number. From the relations involving inversions we obtain that

$$G_{n+1}(q) = G_n(q) + \sum_{k=1}^{n-1} G_k(q)G_{n-k-1}(q)q^{(k+1)(n-k)}, \qquad G_1(q) = 1.$$

By means of the q-Motzkin definition, we get $G_n(q) = M_{n-1}(q)$ and this proves Theorem 5.2. The other relations give the recurrence (5.2). By putting $a = q$ and $b = q^{-1}$, we obtain

$$\hat{M}_{n+1}(s,q,q^{-1}) = \hat{M}_n(sq,q,q^{-1}) + s\sum_{k=0}^{n-1} \hat{M}_k(sq,q,q^{-1})\hat{M}_{n-k}(s,q,q^{-1}).$$

From the q-Motzkin definition we deduce (5.3) and this proves Theorem 5.3.

We can also prove Theorem 5.2 by using the steep staircase polyominoes' results and the following combinatorial property:

Lemma 5.4. *The set of steep Dyck words having length $2n$ and k inversions is in bijection with the set of steep staircase polyominoes having width n and area $n(n+1)/2 - k$.*

Proof. Let w be a steep Dyck word having length $2n$ and k inversions. We examine the set of cells P between the Dyck path of w and the path of $\bar{x}x\bar{x}x\ldots\bar{x}x\bar{x}x$ (see Fig. 10). Since the inversion number of w is equal to the area of the Ferrers diagram lying between the Dyck path of w and the path of $x\ldots x\bar{x}\ldots\bar{x}$, the number of cells of P is $n(n+1)/2 - k$. Moreover, each north-east step (s_{i-1}, s_i) of the Dyck path (i.e. each x of w) corresponds to a column of P having length c_i equal to the ordinate of s_i. But the path is a steep Dyck path and so the length of these columns are such that

$$\text{for any } i \in [1..n],\ c_{i+1} = c_i + 1 \text{ or } c_{i+1} < c_i.$$

Therefore, if we translate P's columns along the southeast direction as shown in Fig. 10, we obtain a steep staircase polyomino having n columns and area $n(n+1)/2 - k$.

The reverse side of the bijection is easily obtained by reversing the construction. □

It follows from Theorem 4.3 that the generating function steep staircase polyominoes according to their width and area is $G_{\text{SS}}(1,x,q) = \sum_{n\geqslant 1} \tilde{M}_{n-1}(q)x^n$, and, by means of

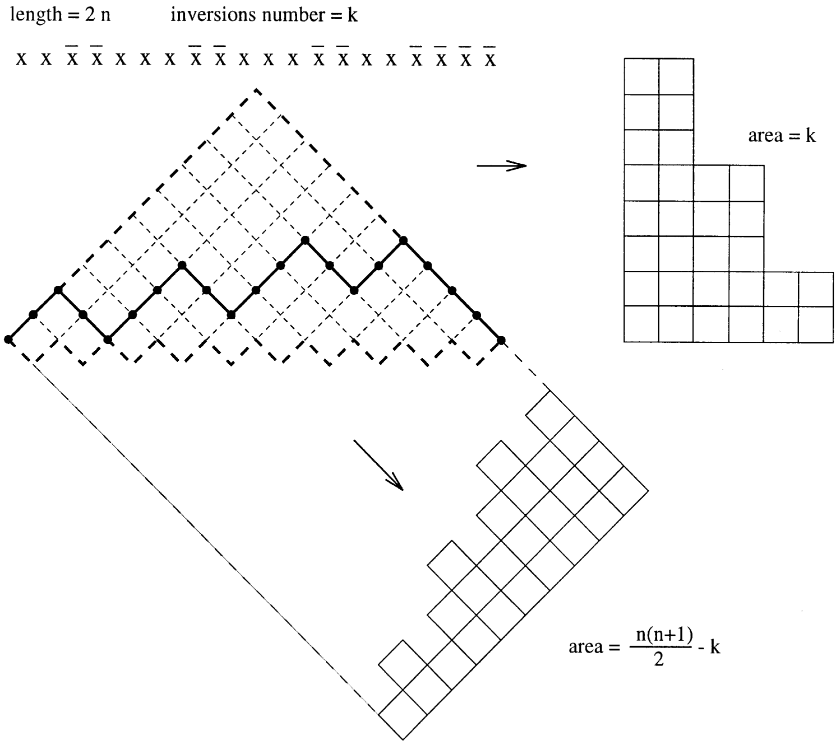

Fig. 10. A bijection between steep Dyck words and steep staircase polyominoes.

Lemma 5.4, we deduce $G_{\mathrm{SW}}(x,q)=\sum_{n\geqslant 1} q^{\binom{n+1}{2}}\tilde{M}_{n-1}(q)\,x^n$. However, $M_n(q)=q^{\binom{n+2}{2}}$ $\tilde{M}_n(q^{-1})$ and this proves Theorem 5.2.

6. Conclusions

We introduced three definitions of the q-analogs of the Motzkin numbers which are similar to the definitions of Carlitz and Polya–Gessel q-Catalan numbers. We illustrated some combinatorial interpretations of these q-Motzkin numbers by means of some recent techniques of counting and q-counting.

An interesting problem regards determining some relations between these q-Motzkin numbers and the Carlitz and Polya–Gessel q-Catalan numbers.

Another interesting problem regards determining some relations between these q-Motzkin numbers and the q-Motzkin numbers derived from several combinatorial statistics over some other classes of combinatorial objects enumerated by Motzkin numbers. For instance, the area of Motzkin paths [5,8,20], the internal path length of some classes of planar trees, such as right-leafed trees and tip-augmented trees [3,17], the inversions number of the class $M_n(321,3\bar{1}42)$ of permutations with forbidden subsequences [4]. Another interesting problem regards determining some relations between these q-Motzkin numbers.

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